Math 7: Handout 11 [2022/12/10] Poker Probabilities

CLASSWORK

In the game of poker, a player is dealt five cards from a regular deck with 4 suits ($\spadesuit, \clubsuit, \diamondsuit, \heartsuit$) with card values in the following order: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A. We calculated probabilities of the following combinations:

Royal Flush: 10, J, Q, K, A of any suit
(Example: 10$\heartsuit$, J$\heartsuit$, Q$\heartsuit$, K$\heartsuit$, A$\heartsuit$)
There are only 4 of them.

Straight Flush: Five cards in a row of the same suit
(Example: 6$\heartsuit$, 7$\heartsuit$, 8$\heartsuit$, 9$\heartsuit$, 10$\heartsuit$)
Each of these can start from any card from 2 to 9, and be in each of the for suits: $8 \times 4 = 32$. Notice that we excluded royal flushes from out computation (if we start with 10, we get a Royal Flush).

Four of a kind: Four cards of the same value, and one other random card
(Example: K$\heartsuit$, K$\spadesuit$, K$\diamondsuit$, K$\clubsuit$, 2$\clubsuit$)
Card value $13\times$ Other card value $12\times$ Other card suit $4 = 13 \cdot 12 \cdot 4$.

Full House: Three cards of the same value (triple), and two cards of the same value(pair)
(Example: K$\heartsuit$, K$\spadesuit$, K$\diamondsuit$, 4$\spadesuit$, 4$\clubsuit$)
Value of the triple $13 \times$ Suits of the triple $\binom{4}{3}\times$ Value of the pair $12\times$ Suits of the pair $\binom{4}{2} = 13\binom{4}{3} \cdot 12\binom{4}{2}$.

Flush: Five cards of the same suit, of any value in any order
(Example: 3$\heartsuit$, 6$\heartsuit$, 8$\heartsuit$, J$\heartsuit$, A$\heartsuit$)
Suit $4\times$ Any five values $\binom{13}{5} = 4\binom{13}{5}$. We also need to exclude Royal Flushes and Straight Flushes, so the total is $4\binom{13}{5} - 36$.

Straight: Sequence of five card values, of any suit
(Example: 5$\heartsuit$, 6$\spadesuit$, 7$\diamondsuit$, 8$\spadesuit$, 9$\clubsuit$)
First card of a sequence (anything from 2 to 10) $9\times$ Five suits $4^5 = 10 \cdot 4^5$. From here we also need to exclude Royal Flushes and Straight Flushes, so the final answer if $10 \cdot 4^5 - 36$.

Triple: Three cards of the same value, and two other random cards
(Example: K$\heartsuit$, K$\spadesuit$, K$\diamondsuit$, 4$\spadesuit$, 2$\clubsuit$)
Value of the triple $\binom{13}{1}\times$ Suits of the triple $\binom{4}{3}\times$ Values of the other two cards $\binom{12}{2}\times$ Suits of the other two cards $4^2 = \binom{13}{1}\binom{4}{3}\binom{12}{2}4^2$.

Two pairs: Two same-value pairs and a random card
(Example: K$\heartsuit$, K$\spadesuit$, 10$\diamondsuit$, 10$\spadesuit$, 4$\clubsuit$)
Two different pair values $\binom{13}{2}\times$ Different suits for each of pair $\binom{4}{2}^2\times$ Remaining value $11\times$ Remaining suit $4 = \binom{13}{2}\binom{4}{2}^2 \cdot 11 \cdot 4$.

Pair: Two cards of the same value, and three other random cards
(Example: K$\heartsuit$, K$\spadesuit$, Q$\diamondsuit$, 4$\spadesuit$, 2$\clubsuit$)
Pair value $\binom{13}{1}\times$ Different suits of the pair $\binom{4}{2}\times$ Three other values $\binom{12}{3}\times$ Three other suits card $4^3 = \binom{13}{1}\binom{4}{2}\binom{12}{3}4^3$.

Anything: To calculate probabilities of each of these combinations, we have to divide the counts above by the total number of poker hands, which is five out of 52 cards $\binom{52}{5} = 2,598,960$ combinations. The table below gives the probabilities and the odds.

Combination	Count	Probability	Odds
Royal Flush	4	0.000154%	1 : 649,740
Straight Flush	32	0.00123%	1 : 81217
Four of a Kind	$13 \cdot 12 \cdot 4$	0.024%	1 : 4,165
Full House	$13\binom{4}{3} \cdot 12\binom{4}{2}$	0.1441%	1 : 693
Flush	$4\binom{13}{5} - 36$	0.1967%	1 : 508
Straight	$10 \cdot 4^5 - 36$	0.3532%	1 : 283
Triple	$\binom{13}{1}\binom{4}{3}\binom{12}{2}4^2$	2.1128%	1 : 46.3
Two Pairs	$\binom{13}{2}\binom{4}{2}^2 \cdot 11 \cdot 4$	4.7539%	1 : 20
Pair	$\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3$	42.2569%	1 : 1.37
Nothing		50.1177%	1 : 0.995

HOMEWORK

1. (a) How many 10-letter "words" one can write using 4 letters H and 6 letters T?

(b) If we toss a coin 10 times and record the result as a sequence of letters H and T (writing H for heads and T for tails), how many different possible sequences we can get? How many of them will have exactly 6 tails?

(c) If we toss a coin 10 times, what are the chances that there will be 6 tails? 3 tails? at least one tails?

2. If we randomly select 100 people form the population of the US, what are the chances that exactly 50 of them will be males? that at least 50 will be males? that all 100 will be males?

3. How many ways are there to divide 12 books *evenly*

(a) Between two bags

(b) Between two bookshelves (order on each bookshelf matters!)

(c) Between three bags

(d) Between three bookshelves (order on each bookshelf matters!)

4. A person is running down the staircase. He is in a rush, so he may jump over some steps. If the staircase is 12 steps (including the top one, where he begins, and the last one, where he ends), in how many ways can he reach the bottom step in 5 jumps? What if there are no restrictions on the number of jumps? [Hint: keep track of the steps he steps on. . .]