Math 7: Handout 9
Binomial coefficients formula. Binomial probabilities

Binomial coefficients. Binomial coefficients provide answers to the following questions:

$$\begin{aligned}\binom{n}{k} &= \text{the number of paths on the chessboard going } k \text{ units up and } n-k \text{ to the right}\\ &= \text{rhe number of words that can be written using } k \text{ ones and } n-k \text{ zeroes}\\ &= \text{the number of ways to choose } k \text{ items out of } n \text{ (order doesn't matter)}\end{aligned}$$

Formula for binomial coefficients. It turns out that there is an explicit formula for $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

Compare it with the number of ways of choosing k items out of n when the order matters:

$${}_nP_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

For example, there are $5 \cdot 4 = 20$ ways to choose to items out of 5 if the order matters, and $\frac{5\cdot 4}{2} = 10$ if the order doesn't matter. In general,

$$\begin{pmatrix}\text{number of ways to choose}\\ k \text{ items out of } n,\\ \text{order matters}\end{pmatrix} = \begin{pmatrix}\text{number of ways to choose}\\ k \text{ items out of } n,\\ \text{order does not matter}\end{pmatrix} \cdot \begin{pmatrix}\text{number of ways to reorder}\\ k \text{ items among themselves}\end{pmatrix}$$

Binomial Probabilities. These numbers are also useful in calculating probabilities. Imagine that we have some event that happens with probability p ("success") and does not happen with probability $q = 1 - p$ ("failure"). Then the probability of getting k successes in n trials is

$$P(k \text{ successes in } n \text{ trials }) = \binom{n}{k} p^k q^{n-k}, \text{ where}$$

- p — probability of success in one try;
- $q = 1 - p$ — probability of failure in one try;
- n — number of trials;
- k — number of successes;
- $n - k$ — number of failures.

Example: Move a figure from lower left corner 8 times one square up or to the right with equal probability. What are the probabilities to end up in any square of the board?
Solution: Total number of steps is $n = 8$, and you make $k_\uparrow + k_\rightarrow = 8$ steps with probability $p = 1/2$ up and $q = 1/2$ down. **Example:** You roll a die 100 times. What is the probability of getting a 6 exactly 20 times?
Solution: Here we have: $n = 100$, $k = 20$, $p = 1/6$, $q = 5/6$. Then

$$P = \binom{100}{20} \cdot \left(\frac{1}{6}\right)^{20} \left(\frac{5}{6}\right)^{80}$$

Example 3: If we draw 3 cards out of the deck, what are the chances that

- They will all be all spades
- They will be all aces
- That they will be ace of spades, queen of spades, and king of spades, in this order
- That they will be queen of spades, ace of spades, and king of spades, in this order
- That they will be ace, queen, and king of spades, in some order

In problems on combinatorics and probability, you can give your answer as a binomial coefficient without calculating it. The answer presented in this way explains your solution and shows your thinking.

Homework

1. There are 15 players in a soccer club.
  (a) How many ways does the coach have to select 11 players for a match?
  (b) How many ways does the coach have to select 10 players and one goalkeeper?

2. In one a lottery "Sweet Million" run by the New York State, 6 numbers are chosen randomly from 1–40. If you guess all 6 correctly (order does not matter), you win $1,000,000. [Let's ignore smaller prizes for guessing 5 out of 6, etc.]
  (a) How many ways are there to choose 6 numbers out of 40?
  (b) What are your chances of winning?
  (c) How much money does New York State make for each ticket sold (on average), if a lottery ticket cost $1?

3. In poker, players are drawing "hands" (combinations of 5 cards) from the 52-card deck (4 suits, 13 cards in each).
  (a) How many possible hands are there?
  (b) What are your chances of drawing a hand with all spades?
  (c) What are your chances of drawing a with 4 queens? [Hint: how many such hands are there?]
  (d) What are your chances of drawing a royal flush (Ace, King, Queen, Jack, 10 of the same suit)? [Hint: what are your chances of drawing a royal flush in a given suit, say spades?]

4. A hunter is shooting ducks. Probability of hitting a duck with one shot is $p = 1/3$. The hunter makes 5 shots. What is the probability that
  (a) What is the probability that she misses all five?
  (b) What is the probability that she will hit a duck at least once? Will this probability double if she makes 10 shots? (You can use the calculator for computing the answers)
  (c) What is the probability that out of 5 shots, she will hit exactly once? Will this probability double is she makes 10 shots?
  (d) What is the probability that out of 5 shots, she will hit a duck exactly three times? Will this probability double if she makes 10 shots? (You can use the calculator for computing the answers)
  (e) What is the probability that she hits a duck half times or more if she fires 5 times (that is, 3, 4, or 5 hits)? What about if she fires 10 times (that is 5, 6, 7, 8, 9, or 10 hits)?
  (f) What is the most likely number of hits out of 5 shots? And out of 10 shots?

5. At a fair, they offer you to play the following game: you are tossing small balls in a large crate full of empty bottles; if at least one of the balls lands inside a bottle, you win a stuffed toy (worth about $5). Unfortunately, it is really impossible to aim, so the game is just a matter of luck (or probability theory): every ball you toss has a 20% probability of landing inside the bottle.
  (a) If you are given three balls, what is the probability that all three will be hits? That all three will be misses? That at least one will be a hit?
  (b) Same questions for five balls.
  (c) What about seven balls?
  (d) How much should the organizers charge for 3 balls to break even? What about for 5 balls?