MATH 7: HANDOUT 8 [2022/11/13]
INTRODUCTIONS TO COMBINATORICS. PASCAL TRIANGLE.

COUNTING

Fundamental Principle of Counting (Multiplication rule). If the first task can be performed in m ways, and for each of these a second task can be performed in n ways, and for each combination a third task can be performed in k ways, etc. then this entire sequence of tasks can be performedin $m \cdot n \cdot k \dots$ ways.

Permutations: the choice of k things from a set of n things without repetition("replacement") and where the **order matters**.

1. Picking first, second, and third place winners from a group. If a group has n members, then this can be done in $n(n-1)(n-2)$ ways.

Permutations of n things: The number of permutations of all n different things: $n!$

1. Arranging/ordering all n members of a group can be done in $n!$ ways.
2. Listing the favorite deserts in the order of choices: if there are n desserts in total, there are $n!$ ways to arrange them in the order of preference.

Combinations: the choice of k things from a set of n things without repetition ("replacement") and where **order does not matter**. Combinations are harder to count: we will talk about it later!

1. Picking three team members from a group (it doesn't matter who is chosen first, or second, or third).
2. Picking two deserts from a tray (the order in which you eat them doesn't matter!).

PASCAL TRIANGLE

How many ways are there to go from the bottom left corner of the chessboard to the upper right, moving always only to the right and up?

To make out life easier, we will refer to cells by two numbers (m, n): m is the number of the column (counting from left), and n is the number of the row (counting from the bottom).

We can solve this problem iteratively. There is only one path to any of the cells in the lowest row or the left column. Let's put ones there. Now, let's think about other cells.

To get to cell $(2, 2)$, we can first get to cell $(2, 1)$ (there's only 1 way to get there), and then do a step up; or first get to cell $(1, 2)$ (there's again only 1 way to get there), and then do a step right. That means, there are $1 + 1 = 2$ ways to get to cell $(2, 2)$.

Now, let's think about cell $(3, 2)$. Again we have two choices: first, we can get there from cell $(2, 2)$ by doing a step right, or second, we can get there from cell $(3, 1)$ by doing a step up. It means the total number of paths to get to cell $(3, 2)$ will be equal to the total number of ways to get to $(2, 2)$ plus the total number of ways to get to $(3, 1)$: $2 + 1 = 3$.

Keeping this process going, we can notice that the number of paths to cell (m, n) is equal to the number of paths to cell $(m-1, n)$ plus the number of paths to cell $(m, n-1)$. This way we can fill out the entire table:

1	6	21	56	126	252
1	5	15	35	70	126
1	4	10	20	35	56
1	3	6	10	15	21
1	2	3	4	5	6
1	1	1	1	1	1

These numbers are called the *binomial coefficients*. They are usually usually written in a slightly different way:

$$
\begin{array}{c}
1\\
1 \quad 1\\
1 \quad 2 \quad 1\\
1 \quad 3 \quad 3 \quad 1\\
1 \quad 4 \quad 6 \quad 4 \quad 1\\
\dots
\end{array}
$$

This triangle is called **Pascal triangle**. Every entry in it is obtained as the sum of two entries above it. The k-th entry in n-th line is denoted by $\binom{n}{k}$, or by ${}_nC_k$. Note that both n and k are counted from 0, not from 1: for example, $\binom{2}{1} = 2$ or $\binom{3}{0} = 1$.

Classwork

1. For an introductory English class, the professor may choose one out of three novels, and two out of five specified plays. How many different reading lists could a professor create within these parameters?
2. There are 9 problems in this homework. If you only have time to do 8 of them, in how many ways can you choose them?
3. How many ways are there to put 8 rooks on a the chessboard so that no one attacks the others?
4. A dressmaker has two display windows, one for regular dresses and one for evening dresses. If she has 5 regular and 10 evening dresses, how many display arrangements can she make?

HOMEWORK

1. Using algebraic identities for $(a-b)^2$, $(a+b)^2$, a^2-b^2, calculate

(a) $101^2 - 202 + 1 =$

(b) $501^2 =$

(c) $48 \cdot 52 =$

(d) $998^2 =$

2. Simplify these expressions with radicals (so that there are no radicals in the denominators):

(a) $\frac{1}{\sqrt{5}-\sqrt{3}}$

(b) $\frac{1}{\sqrt{3}} + \frac{1}{2-\sqrt{3}}$

(c) $\frac{2}{3-\sqrt{7}}$

(d) $\frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1}$

3. A dinner in a restaurant consists of 3 courses: appetizer, main course, and dessert. There are 5 possible appetizers, 6 main courses and 3 desserts. How many possible dinners are there?

4. How many ways are there to seat 5 students in a class that has 5 desks? if there are 10 desks?

5. How many ways are there to select first, second and third prize winner if there are 14 athletes in a competition?

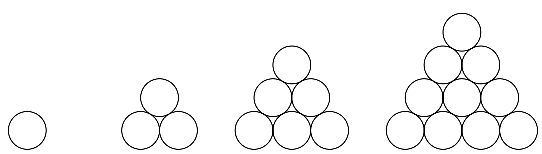
6. Finish the chessboard problem (for 8×8-board: how many ways are there to go from lower left corner to upper right corner)?

7. What is the sum of all entries in the n-th row of Pascal triangle? Try computing first several answers and then guess the general formula.

8. What is the alternating sum of all the numbers in nth row of Pascal triangle, i.e.

$$1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots$$

9. Let us draw a figure consisting of n rows of circles as shown in the figure below (for $n = 1, 2, 3, 4$):



Let T_n be the number of circles in n-th figure (for example, $T_1 = 1$, $T_2 = 3$, $T_3 = 6\dots$). These numbers are sometimes called the **triangular numbers**.

(a) What is the difference $T_{n+1} - T_n$?

(b) Show that the numbers T_n appear in the Pascal triangle as shown below

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & \boxed{1} & & \\
 & 1 & & 3 & & \boxed{3} & & 1 & \\
1 & & 4 & & \boxed{6} & & 4 & & 1 \\
 & & & & \dots & & & &
\end{array}$$

that is, $T_n = \binom{n+1}{2}$.