MATH 7: HOMEWORK 20
COORDINATE GEOMETRY: HYPERBOLAS AND PARABOLAS

Review of Quadratic Equations

Here is what we have learned so far about quadratic equations:

- A **quadratic polynomial** is an expression of the form $p(x) = ax^2 + bx + c$.
- **Roots** of a quadratic polynomial are numbers such that $p(x) = 0$. If x_1, x_2 are roots, then $p(x) = a(x - x_1)(x - x_2)$.
- **Vietá formulas**: If x_1, x_2 are roots of $x^2 + bx + c$, then

$$x_1 + x_2 = -b$$
$$x_1 x_2 = c$$

- **Completing the square**: we can rewrite

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a} = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right) \tag{1}$$

where $D = b^2 - 4ac$.

From this, one gets the **quadratic formula**: if $D < 0$, there are no roots; if $D \geq 0$, then the roots are

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \tag{2}$$

- From formula (1), we see that:
  - If $a > 0$, then the **smallest** possible value of $p(x)$ is $-\frac{D}{4a}$, which happens when $x = -\frac{b}{2a}$. In this case the graph is a parabola with branches going up.
  - If $a < 0$, then the **largest** possible value of $p(x)$ is $-\frac{D}{4a}$, which happens when $x = -\frac{b}{2a}$. In this case the graph is a parabola with branches going down.

Graphs of quadratic functions

- We know how to draw the graph of $y = x^2$. It's a parabola.
- We know that the graph of $y = x^2 + b$ can be obtained from the graph of $y = x^2$ by shifting up by b units (or down, if $b < 0$)
- We know that the graph of $y = (x + a)^2$ can be obtained from the graph of $y = x^2$ by shifting *left* by a units (or right, if $a < 0$).
- Based on the two fact above, we can draw a graph of any function of the type $y = (x + a)^2 + b$.

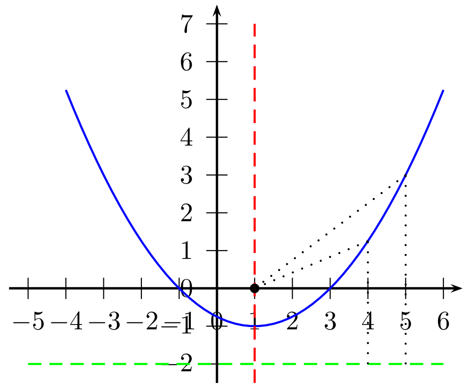
We can transform any quadratic function $y = x^2 + px + q$ to $y = (x + a)^2 + b$ by completing the square.

Properties of a Parabola

A parabola is the set of all points in a plane that are equally distant away from a given point and a given line (see black dotted lines).

This given point is called the **focus** (black dot) of the parabola and the line is called the **directrix** (green line).

If the parabola is of the form $(x-h)^2 = 4p(y-k)$, the vertex is $(h.k)$, the focus is $(h, k+p)$ and directrix is $y = k-p$



HOMEWORK

1. For what values of a does the polynomial $x^2 + ax + 14$ has no roots? exactly one root? two roots?

2. Let x_1, x_2 be the roots of the equation $x^2 + 3x + 4 = 0$. Without calculating the roots, find:
   (a) $x_1^2 + x_2^2$
   (b) $\frac{1}{x_1^2} + \frac{1}{x_2^2}$

3. A circle with center $(3, 5)$ intersects the y-axis at $(0, 1)$.
   (a) Find the radius of the circle
   (b) Find the coordinates of the other point of intersection on the y-axis
   (c) What are the coordinates of the intersection points of the circle with the x-axis?

4. Of all the rectangles with perimeter 4, which one has the largest area?
[**Hint:** if sides of the rectangle are a and b, then the area is $A = ab$, and the perimeter is $2a + 2b = 4$. Thus, $b = 2 - a$, so one can write A using only $a\ldots$]

5. Prove that for any point P on the parabola $y = \frac{x^2}{4} + 1$, the distance from P to the x-axis is equal to the distance from P to the point $(0, 2)$.

6. Use completing the square method to draw the following graphs:

   (a) $y = x^2 - 5x + 5$
   (b) $y = x^2 - 4x + 2$
   (c) $y = x^2 - x - 1$
   (d) $y = -x^2 + 3x - 0.5$
   (e) $y = x^2 + 4x - 4$

7. Graph $y = (\sqrt{x})^2$. Note $x \geq 0$

8. A triangle ABC has corners $A(-3, 0)$, $B(0, 3)$ and $(3, 0)$. The line $y = \dfrac{1}{3}x + 1$ separates the triangle in 2. What is the area of the piece lying below the line?

9. Sketch graphs of the following functions:

   (a) $y = (x - 1)^2 + 1$
   (b) $y = \dfrac{1}{x + 2} + 1$
   (c) $y = \dfrac{1}{2 - x}$
   (d) $y = \dfrac{x + 2}{x + 1}$
   (e) $y = \left|\dfrac{1}{x - 1} + 1\right|$