MATH 7: HOMEWORK 15
Inequalities
February 5, 2023

1. Linear inequalities
Linear inequalities are like linear equations: they have terms with x and numbers only. When solving linear inequalities, we use the same techniques as those used for solving linear equations. The important exception to this is that when multiplying or dividing by a negative number, you must reverse the direction of the inequality.
Example 1. $x + 2 \le 5$
Solution: Subtracting 2 from both sides, we get $x \le 3$. Therefore, any value of x less than or equal to 3 would make this inequality true. We can write this solution using interval notation: $(-\infty, 3]$.
Notice, that square bracket represents that the ends of intervals that are included. If we don't want to include the end of the interval, we would use round parentheses.

2. Product of two expressions. Quadratic inequalities
If we have an inequality that involves a product of two expressions ***compared to 0***, we should consider several cases:
- If the inequality is $f(x) \cdot g(x) > 0$, where $f(x)$ and $g(x)$ are some expressions involving x, then there are two cases:
  1) both $f(x) > 0$ **and** $g(x) > 0$ **,or** 2) both $f(x) < 0$ **and** $g(x) < 0$.
- If the inequality $if\ f(x) \cdot g(x) < 0$, where $f(x)$ and $g(x)$ are some expressions involving x, then there are two cases:
  1) $f(x) < 0$ **and** $g(x) > 0$ **,or** 2) $f(x) > 0$ **and** $g(x) < 0$.

Example 2. $(x - 1)(x + 2) \ge 0$.
Solution: A product of two expressions is positive in two cases: 1) when both expressions are positive; or when 2) both expressions are negative. Let us consider them separately:
Case 1: $x - 1 \le 0$ and $x + 2 \le 0$. This comes down to two inequalities: $x \le 1$; $x \le -2$. For both of them to be true, x should be less than -2 on the number line: $x \le -2$, or $(-\infty, -2]$ in interval notation..
Case 2: $x - 1 \ge 0$ and $x + 2 \ge 0$. This comes down to two inequalities: $x \ge 1$; $x \ge -2$. For both of them to be true, x should be above 1 on the number line: $x \ge 1$, or $[1, \infty)$ in interval notation.
The final solution is the combined solution of Case 1 OR Case 2: $x \le -2$ or $x \ge 1$. We can use a union of intervals to write the solution: $(-\infty, -2] \cup [1, \infty)$.

3. Rational Inequalities
A similar approach that we discussed in the previous section works for fractions of two expressions involving x.
Example 3. $\frac{x-4}{2x-10} \le 0.$

Solution: Notice that we can't multiply both parts by $2x - 10$, since we don't know whether the expression it's positive or negative — multiplication by a negative expression would change the sign, so we should be careful. We will use a different method. A fraction is negative in two cases: 1) when the numerator is positive, and the denominator is negative; 2) when the numerator is negative, and the denominator is positive. Let us consider them separately:
Case 1: $x - 4 \ge 0$; and $2x - 10 < 0$. This comes down to two inequalities: $x \ge 4$; $x < 5$. For both of them to be true, x should be between 4 and 5: $4 \le x < 5$.
Case 2: $x - 4 \le 0$; and $2x - 10 > 0$. This comes down to two inequalities $x \le 4$; $x > 5$. Both of them cannot be true simultaneously, so this case is impossible.
The final solution is $4 \le x < 5$, or $[4, 5)$.

Example 4. $\frac{2x-1}{x+3} \geq 3.$

Solution: Notice that as before, we can't multiply both parts by $x + 3$, since we don't know whether it's positive or negative — multiplication by a negative expression would change the sign, so we should be careful. Therefore, we will subtract 3 from both sides:

$$\frac{2x-1}{x+3} \geq 3$$

$$\frac{2x-1}{x+3} - 3 \geq 0$$

$$\frac{(2x-1) - 3(x+3)}{x+3} \geq 0$$

$$\frac{-x-10}{x+3} \geq 0$$

Now, a fraction is positive in two cases: 1) numerator is positive, denominator is positive; 2) numerator is negative, denominator is negative. Let us consider them separately:

Case 1: $-x - 10 \geq 0$; and $x + 3 > 0$. This comes down to two inequalities: $x \leq -10$; $x > -3$. Both of them cannot be true simultaneously, so this case is impossible.

Case 2: $-x - 10 \leq 0$; and $x + 3 < 0$. This comes down to two inequalities $x \geq -10$; $x < -3$. For both of them to be true, x should be between -10 and -3: this is written as $-10 \leq x < -3$, or $[-10, -3)$.

The final solution is $-10 \leq x < -3$, or $[-10, -3)$.

4. Inequalities with parameter

Sometimes you will have to solve inequalities depending on the value of a parameter $\boldsymbol{a}$. Make sure to consider various cases when dividing/multiplying by $\boldsymbol{a}$ or any expression involving a, since it could be positive or negative!

Example 5. Solve the following inequality depending on the value of a: $ax - 5 < 0$.

Solution: This inequality is equivalent to $ax < 5$. Now, we have three cases:

Case 1: If $a = 0$, then any value of x satisfies the equation.

Case 2: If $a > 0$, then the solution is $x < 5/a$.

Case 3: If $a < 0$, then the solution is $x > 5/a$, since dividing by negative a changes the sign of the inequality.

Example 6. Solve the following inequality depending on the value of a: $x^2 > a$.

Solution: We will consider two cases: $a \leq 0$ and $a > 0$:

Case 1: If $a \leq 0$, then any value of x satisfies the equation.

Case 2: If $a > 0$, then we get $x > \sqrt{a}$ or $x < -\sqrt{a}$, or written as intervals $(-\infty, -\sqrt{a}) \cup (\sqrt{a}, \infty)$

Homework problems

1. Solve the following linear inequalities:
   a. $4x + 6 < 2x + 14$
   b. $-2(x + 3) < 10$
   c. $-2(x + 2) > 4 - x$

2. Solve the following rational inequalities:
   a. $\frac{1+3x}{x-2} > 0$
   b. $\frac{x+1}{x-5} \leq 0$
   c. $\frac{3x+1}{x+4} \geq 1$

3. Find all values of $\boldsymbol{a}$ so that the following equation has a positive solution. [Hint: Express (solve for) x in terms of $\boldsymbol{a}$, and then find the values of $\boldsymbol{a}$ so that the resulting expression is positive]

$$4 - a = \frac{2}{x - 1}$$

4. Solve the following inequalities:
   a. $x^2 \leq 81$
   b. $x^2 \geq 100$
   c. $\sqrt{x} \leq 7$
   d. $\sqrt{x + 10} < 2$

5. Depending on the value of a, solve the following inequality: $(a - 2)x > a^2 - 4$.

6. Depending on the value of a, solve the following inequality: $\sqrt{x} \geq a$.