MATH 7 HOMEWORK 6: Geometric sequences
October 30, 2022

1. Geometric sequence (progression)

A sequence of numbers is a geometric progression if the next number in the sequence is the current number times a constant called the **common ratio**, let's call it **q**.

For example: 6, 12, 24, 48, ... The common ratio here is q = 2.

Sequence elements (terms) are labeled according to their position in the sequence using a counter **n** as a subscript. The value of the n-th element in a sequence is labeled as a_n. Then, the first term in the sequence has n = 1 and a value of $a_1 = 6$, the second element is $a_2 = 15$, and so on.

We could find any element of a sequence knowing the first element a_1 and the ration q.
Example: *What is b_{10}? What is the n^{th} term?*

$a_1 = 6$

$a_2 = a_1 \times q = 6 \times 2 = 12$

$a_3 = a_2 \times q = (a_1 \times q) \times q = a_1 \times q^2 = 6 \times 2^2 = 24$

$a_4 = a_3 \times q = (a_1 \times q^2) \times q^2 = a_1 \times q^3 = 6 \times 2^3 = 48$

....

$\boldsymbol{a_n = a_1 \times q^{n-1}}$

So $a_{10} = a_1 \times q^9 = 6 \times 2^9 = 6 \times 512 = 3072$

2. Property of a geometric sequence

A property of a geometric sequence is that any term is geometric mean of its neighbors or any two equally distanced neighbors.

$$a_n = \sqrt{a_{n-1}.a_{n+1}} = \sqrt{a_{n-k}.a_{n+k}}$$

Proof:

$$a_n = a_{n-1} \times q$$

$$a_n = a_{n+1} \div q$$

Multiplying these two equalities gives us:

$$a_n^2 = a_{n-1} \cdot a_{n+1}$$

from where we can get what we need.

3. Sum of a geometric sequence,

a) Sum of the first n-terms:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = a_1 \times \frac{(1-q^n)}{1-q}$$

Proof: To prove this, we write the sum and we multiply it by q:

$$S = a_1 + a_2 + a_3 + \cdots \mathrm{a}_{\mathrm{n-1}} + a_n$$
$$qS = qa_1 + \mathrm{q}a_2 + \mathrm{q}a_3 + \cdots + qa_{n-1} + qa_n$$

Remember that $qa_{n-1} = a_n$, so that the last term is $qa_n = q \times (a_1 \times q^{n-1}) = a_1 \times q^n$:

$$S = a_1 + a_2 + a_3 + \cdots \mathrm{a}_{\mathrm{n-1}} + a_n$$
$$qS = a_2 + a_3 + a_4 + \cdots + a_n + a_{n+1}$$

Subtracting the second equality from the first, and canceling out the terms, we get:

$$S_n - qS_n = (a_1 - a_{n+1}), \quad \text{or}$$

$$S_n(1-q) = (a_1 - a_1 q^n)$$

$$S_n(1-q) = a_1(1-q^n)$$

from which we get the formula above.

b) Sum of **Infinite Sum**

If $0 < q < 1$, then the sum of the geometric progression is approaching some numbers, which we can call a **sum of an infinite geometric progression**, or just an **infinite sum**.

For example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ \ldots \ = \ 2$$

The formula for the infinite sum is the following: $\mathrm{S} \ = \ \frac{a_1}{1-q}$

4. Geometric sequences -summary

$$a_n = a_1 \times q^{n-1}$$

$$a_n = \sqrt{a_{n-1} . a_{n+1}}$$

$$S_n = a_1 \times \frac{(1-q^n)}{1-q}$$

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations **how you arrived at this answer.**

1. Write the first 5 terms of a geometric progression if $a_1 = -20$ and $q = \frac{1}{2}$
2. What are the first 2 terms of the geometric progression: $a_1, a_2, 24, 36, 54, ...$?
3. What is the common ratio of the geometric progression: $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, ...$? What is a_{10}? What is a_{100}?
4. Calculate the sum: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{10}}$
5. What is the sum of : $1 - 2 + 2^2 - 2^3 + 2^4 - 2^5 + \cdots - 2^{15}$?
6. What is the sum of: $1 + x + x^2 + x^3 + x^4 + x^5 + \cdots + x^{100}$?
7. A geometric progression has 99 terms, the first term is 12 and the last term is 48. What is the 50-th term?
8. If we put one grain of wheat on the first square of the chessboard, two on the second, then four, eight,. . . , ***approximately*** how many grains of wheat will there be? (You can use $2^{10} = 1024 \approx 10^3$).
   Can you estimate the total volume of all this wheat? Compare with the annual wheat harvest of the US, which is about 2 billion bushels. (A grain of wheat is about 10 mm^3; a bushel is about 35 liters, or 0.035 m^3)
9. Musicians use special notations for notes, i.e. sound frequencies. Namely, they go as follows:

   . . . , A, A ♯ , B, C, C ♯ , D, D ♯ , E, F, F ♯ , G, G ♯ , A, A ♯ , . . .

   The interval between two notes in this list is called a **halftone**; the interval between A and the next A (or B and next B, etc.) is called an **octave**. Thus, one octave is 12 halftones. (If you have never seen it, read the description of how it works in Wikipedia.)

   It turns out that the frequencies of the notes above form a geometric (not an arithmetic!!) sequence: if the frequency, say, of A in one octave is 440 Hz, then the frequency of A ♯ is 440$\boldsymbol{r}$, frequency of B is 440$\boldsymbol{r}^2$, and so on.
   a. It is known that moving by one octave doubles the frequency: if the frequency of A in one octave is 440 Hz, then the frequency of A in the next octave is 2×440 = 880 Hz. Based on that, can you find the common ratio ***r*** of this geometric sequence?
   b. Using the calculator, find the ratio of frequencies of A and E (such an interval is called a **fifth**). How close is it to 3 : 2?

   <u>Historic reference:</u> the above convention for note frequencies is known as "equal temperament" and was first invented around 1585. However, it was not universally adopted until the beginning of 19th century. One of the early adopters of this tuning method was J.-S. Bach, who composed in 1722–1742 a collection of 48 piano pieces for so tuned instruments, called Well-Tempered Clavier. Find them and enjoy! If you want to know how musical instruments were tuned before that, do your own research.

10. Plot/sketch on the same graph using colored pencils: $y = -x^2$, $y = -(x+3)^2$, $y = -(x+3)^2 - 1$. As in class, your base parabola $y = -x^2$ is being moved along x and y.