Classwork 10.
Review of combinatorics.

1. There are 25 rows in a theater, 20 seats in each raw.
   a. If all tickets are sold, how many different ways are there for all people to seat?
   b. If only 2 tickets are sold?
   c. Only 10?
   d. Only 100?

First, we have find the number of seats: $25 \cdot 20 = 500$. First person will have to chose from 500 possible seats, second will choose from 499 and so on.

$$500 \cdot 499 \cdot 498 \cdot \ldots \cdot 2 \cdot 1 = 500!$$

If 2 ticket sold, there will be:

$$500 \cdot 499 = 500 \cdot (500-1) = 500 \cdot (500-2+1)$$

If 10 tickets are sold:

$$500 \cdot 499 \cdot 498 \cdot 497 \cdot 496 \cdot 495 \cdot 494 \cdot 493 \cdot 492 \cdot 491 =$$
$$= 500 \cdot (500-1) \cdot (500-2) \cdot (500-3) \cdot (500-4) \cdot (500-5) \cdot (500-6) \cdot (500-7) \cdot (500-8) \cdot (500-9)$$
$$= 500 \cdot (500-1) \cdot (500-2) \cdot \ldots \cdot (500-10+2) \cdot (500-10+1)$$

If 100 tickets are sold:

$$500 \cdot (500-1) \cdot (500-2) \cdot \ldots \cdot (500-99)$$

Can we rewrite it in a much shorter way using factorials, just have to multiply by the rest of 500! Factors and then divide by them too, to keep the number the same

$500 \cdot (500-1) \cdot (500-2) \cdot \ldots \cdot (500-99) =$

$$= \frac{500 \cdot (500-1) \cdot (500-2) \cdot \ldots \cdot (500-99) \cdot (500-100) \cdot (500-101) \cdot \ldots \cdot 3 \cdot 2 \cdot 1}{(500-100) \cdot (500-101) \cdot \ldots \cdot 3 \cdot 2 \cdot 1} =$$

$$= \frac{500!}{(500-100)!}$$

As you can see, for 2 tickets we can use the same pattern:

$$500 \cdot 499 = \frac{500!}{498!} = \frac{500 \cdot 499 \cdot 498 \cdot 497 \cdot \ldots \cdot 1}{498 \cdot 497 \cdot \ldots \cdot 1}$$

What about all sold ticket?
We have to end up with :

$$\frac{500!}{(500-500)!}$$

And 500-500 is 0. For the consistency, $0! = 1$ and we are getting our

$$\frac{500!}{(500-500)!} = 500!$$

So, we can write a general expression for calculation a number of the ways (permutations without repetition, object is not going back to the pool of objects) to choose *m* objects out of *n* objects, if the order of the objects in a group is important.

$$P(n,m) = \frac{m!}{(m-n)!}$$

For example, how many 4-digit numbers can be created if no same digits can be used in one number?

We have 10 possible digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Number can't start with 0, so let's count 3-digits "numbers", using 0 for all possible positions:

$$P(10,3) = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10 \cdot 9 \cdot 8$$

And the result should be multiplied by 9 possibilities for the first digit

$$10 \cdot 9 \cdot 8 \cdot 9 = 6480$$

If repetition is allowed, the calculation will be a little different. How many 4-digit numbers can be created out of 10 digits, if we can use repetitive digits, numbers as 2233 is allowed.

There are 10 possibilities for each of place's values starting from hundreds, and 9 possibilities for the first digits:

$$9 \cdot 10 \cdot 10 \cdot 10 = 9000$$

Situation becomes more difficult if order of objects if not important anymore.

We need to choose a team (3) of students to go to math count competition out of 20. All students are equally good at math, we will choose randomly. We know how to calculate the number of permutations of 3 out of 20:

$$P(20,3) = \frac{20!}{(20-3)!} = \frac{20!}{17!} = 20 \cdot 19 \cdot 18 = 6840$$

For each three students, say Kate, Mitchel, and Alice, we have 6 possible permutations between them, 3!:

Kate	Mitchel	Alice
Kate	Alice	Mitchel
Mitchel	Alice	Kate
Mitchel	Kate	Alice
Alice	Mitchel	Kate
Alice	Kate	Mitchel

All these teams were calculated as different teams, but in reality all six of them are the same.

So, we have to divide our final $P(3, 20)$ by 3!

$$C(20,3) = \frac{20!}{(20-3)! \cdot 3!} = \frac{6840}{6} = 1140$$

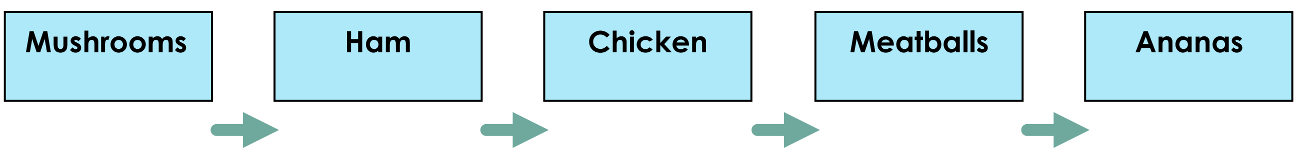
Now we have 1140 possible combinations (not permutations) to choose 3 student team from 20. General formula for combinations (repetition is not allowed) is

$$C(n,m) = \frac{n!}{(n-m)!\,m!}$$

$m!$ is a number of permutations inside of group of m objects, wa have to divide by that, when order does not metter.

The most difficult problem if we need to choose number of objects, order doesn't metter and repetition is allowed.

As an example, we can calculate how many different ways are there to choose 3 toppings from 5 different topics, repetition s allowed, you can take three mushrooms toppings, or mushroom, ham, and chicken. Let take a look on the problem as we are moving from fist container to fifth, choosing toppings.

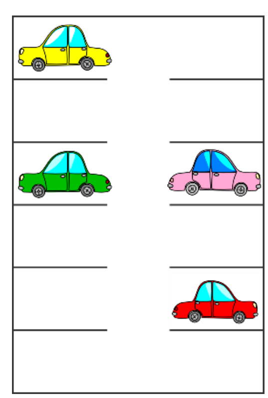


This picture illustrates the three toppings, one mushroom topping and two ham. Problem now is about rearrangement of three circles (number of toppings) and four arrows (number of moves from first container to fifth). We are choosing from $3 + 4 = 3 + (5 - 1)$

$$C_r(5,3) = \frac{\big(3+(5-1)\big)!}{3! \cdot (3+(5-1)-3)!} = \frac{\big(3+(5-1)\big)!}{3!\,(5-1)!} = \frac{7!}{3! \cdot 4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 35$$

For general expression we can write:

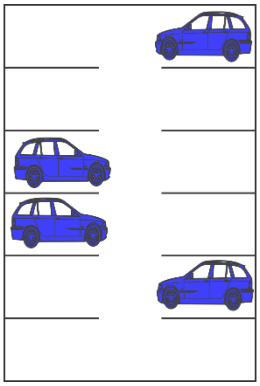
$$C_r(n,m) = \frac{(m+n-1)!}{m!\,(n-1)!}$$



Exercises:

1. Appartment bulding has 12 appartments and a parking for 12 cars (each family has different car). How many different way are there to park these 12 cars?

2. Today there were only 4 cars at the parking lot. How many different ways are there to park 4 cars on a 12-place parking lot?



3. There is a 12-place parking lot at the dealership. 4 identical cars are parked there. How many different way are there to park these 4 cars?

4. There are N boys and N girls on the dance floor. In how many ways can they pair up for the next dance?

5. On a plane 99 points are marked on a straight line and 1 point is marked not on a line. How many different triangles can be drawn?

6. There are 100 points marked on a plane, any three of them do not belongs to the same line. How many triangles can be drawn?

7. The sports club has 30 people, of which four people must be allocated to run the 1000 meters race. In how many ways can this be done?

8. In how many ways can a team of four be formed to participate in the 100m + 200m + 300m + 400m relay?

9. There are three cities in Wonderland A , B and C . There are 6 roads from city A to city B , and 4 roads from city B to city C.
   In how many ways can you travel from A to C ?

10. In Wonderland, they built another city D *and* several new roads - two from A to D and two from D to C.
    In how many ways can you now get from city A to city C ?