Math 5b. Classwork 2.

Exponent. Review.

Exponentiation is a mathematical operation, written as a^n, involving two numbers, the base *a* and the exponent *n*. When *n* is a positive integer, exponentiation corresponds to repeated multiplication of the base: that is, a^n is the product of multiplying *n* bases:

$$a^n = \underbrace{a \cdot a \cdot a \dots \cdot a}_{n\ times}$$

In that case, a^n is called the *n*-th power of *a*, or *a* raised to the power *n*.
The exponent indicates how many copies of the base are multiplied together.

Properties of exponent:

$$a^n \cdot a^m = \underbrace{a \cdot a \dots \cdot a}_{n\ times} \cdot \underbrace{a \cdot a \dots \cdot a}_{m\ times} = \underbrace{a \cdot a \cdot a \dots \cdot a}_{n+m\ times} = a^{n+m}$$

$$(a^n)^m = \underbrace{a^n \cdot a^n \cdot \dots \cdot a^n}_{m\ times} = \underbrace{\underbrace{a \cdot a \cdot \dots \cdot a}_{n\ times} \cdot \dots \cdot \underbrace{a \cdot a \cdot \dots \cdot a}_{n\ times}}_{m\ times} = a^{n \cdot m}$$

$$a^n \cdot a = \underbrace{a \cdot a \cdot a \dots \cdot a}_{n\ times} \cdot a = \underbrace{a \cdot a \cdot a \cdot a \dots \cdot a}_{n+1\ times} = a^{n+1} = a^n \cdot a^1$$

1. $a^n = \underbrace{a \cdot a \cdot a \dots \cdot a}_{n\ times}$
2. $a^n \cdot a^m = a^{n+m}$
3. $(a^n)^m = a^{n \cdot m}$
4. $a^1 = a$, for any a
5. $a^0 = 1$, for any $a \neq 0$
6. $(a \cdot b)^n = a^n \cdot b^n$
7. $a^{-n} = \frac{1}{a^n}$

In order to have the set of power properties consistent, $a^1 = a$ for any number *a*.

$$a^n = a^n \cdot 1 = a^{n+0} = a^n \cdot a^0$$

In order to have the set of properties of exponent consistent, $a^0 = 1$ for any number *a,* but 0.
Also, if there are two numbers *a* and *b*:

$$(a \cdot b)^n = \underbrace{(a \cdot b) \cdot \dots \cdot (a \cdot b)}_{n\ times} = \underbrace{a \cdot \dots \cdot a}_{n\ times} \cdot \underbrace{b \cdot \dots \cdot b}_{n\ times} = a^n \cdot b^n$$

- A positive number raised into any power will result a positive number.
- A negative number, raised in a power, represented by an even number is positive, represented by an odd number is negative.

If a number a in a power n is divided by the same number in a power m,

$$\frac{a^n}{a^m} = \frac{\overbrace{a \cdot a \cdot \ldots a}^{n\ times}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}} = a^{n-m} = \overbrace{a^1 \cdot a^1 \cdot \ldots \cdot a^1}^{n\ times} \cdot \overbrace{a^{-1} \cdot a^{-1} \cdot \ldots \cdot a^{-1}}^{m\ times}$$

$$a^{-1} = \frac{1}{a^1}; \quad a^{-n} = \frac{1}{a^n};$$

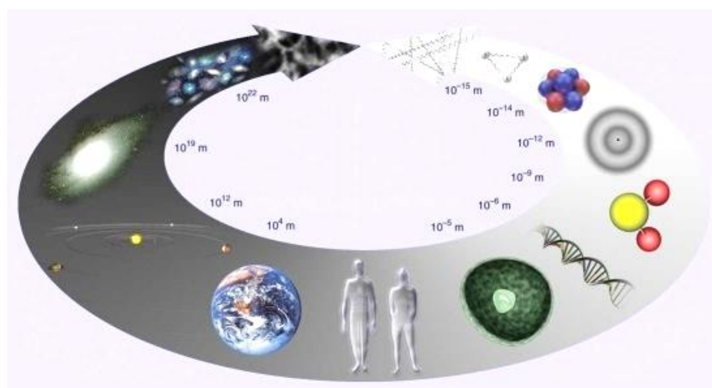
Let’s see how our decimal system of writing numbers works when we use the concept of exponent:

$3456 = 1000 \cdot 3 + 100 \cdot 4 + 10 \cdot 5 + 1 \cdot 6 = 10^3 \cdot 3 + 10^2 \cdot 4 + 10^1 \cdot 5 + 10^0 \cdot 6$

The value of a place of a digit is defined by a power of 10 multiplied by the digit. Very large numbers can be written using this system, as well as very small numbers.

$0.3 = \frac{1}{10} \cdot 3 = 10^{-1} \cdot 3;$

$0.456 = \frac{1}{10} \cdot 4 + \frac{1}{100} \cdot 5 + \frac{1}{1000} \cdot 6 = 10^{-1} \cdot 4 + 10^{-2} \cdot 5 + 10^{-3} \cdot 6$



Scientists work with very large and very small things, from galaxies to viruses. They need to be able to write numbers, describing the object of interest, for example the distance between two galaxies or the diameter of a virus.

One of the most important numbers in the universe is the speed of light.

299 792 458 m / s. It’s very convenient to represent it as a decimal starting with units and multiplied by a power of 10.

$299\ 792\ 458\ m\ per\ s = 2.99792458 \cdot 10^8\ m\ p\ s.$

Then we can round it up (usually to two digits after the point).

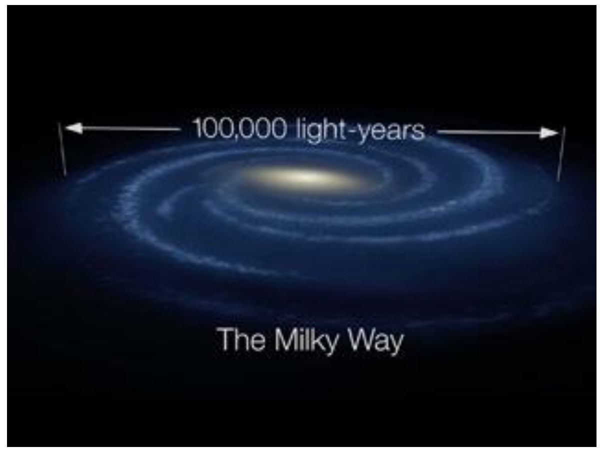
$2.99792458 \cdot 10^8\ m\ p\ s. \approx 3 \cdot 10^8 m\ p\ s.$

Let's convert the value to kilometers per hour. Each kilometer is 1000 meters, so we need to divide it by 1000:

$$3 \cdot \frac{10^8}{10^3} = 3 \cdot 10^8 \cdot 10^{-3} = 3 \cdot 10^{8-3} = 3 \cdot 10^5 m\,p\,s$$

In each hour there are 3600 seconds, or $3.6 \cdot 10^3$ seconds. To find out the speed of light in km per hour we now need to multiply the speed in seconds by $3.6 \cdot 10^3$

$$3 \cdot 10^5 m\,p\,s \cdot 3.6 \cdot 10^3 = 10.8 \cdot 10^8 km\,p\,h$$



The Milky Way galaxy has a diameter of 105,700 light years, so the light will travel from one end to the other through its center in 105700 years.

How far is one side from the other in the Milky Way in kilometers?

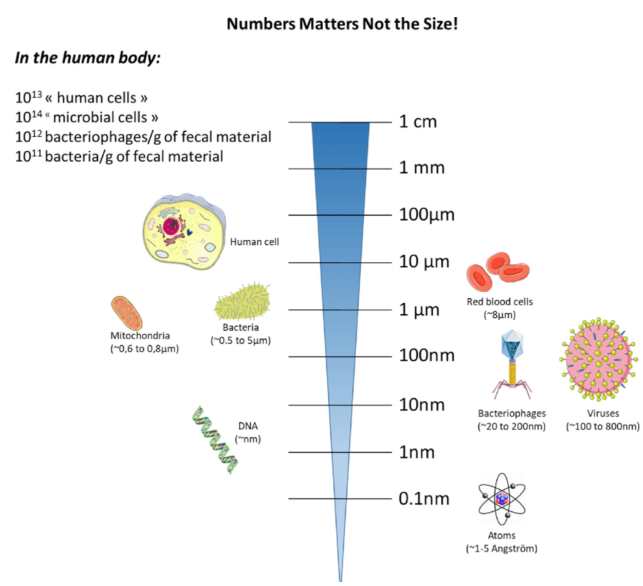
$$10.8 \cdot 10^8 km\,p\,h \ \cdot 105700\ years$$

How many hours in a year? $24 \cdot 365.25 = 8766 \approx 8.8 \cdot 10^3\ hours$

$$10.8 \cdot 10^8 km\,p\,h \ \cdot 105700\ years \approx 10.8 \cdot 10^8 km\,p\,h \cdot 8.8 \cdot 10^3 hours \approx 95.04 \cdot 10^{11} km$$

$$\approx 9.5 \cdot 10^{12}\ km \approx 10^{13} km$$

This way to write numbers is called scientific notation, it's used a lot in science for describing various objects and processes. Let's take a look on the small things, like bacteria and viruses.



$$1\ cm = 0.01\ m = \frac{1}{100} m = \frac{1}{10^2} m = 10^{-2} m$$

$$1\ mm = 0.001\ m = \frac{1}{1000} m = \frac{1}{10^3} m = 10^{-3} m$$

$1\ \mu m = 10^{-6} m; \qquad 1nm = 10^{-9} m$

Bactria are between $0.5 - 1.5\mu m$;
$0.5\ \mu m = 0.5 \cdot 10^{-6} m = 5 \cdot 10^{-7}$

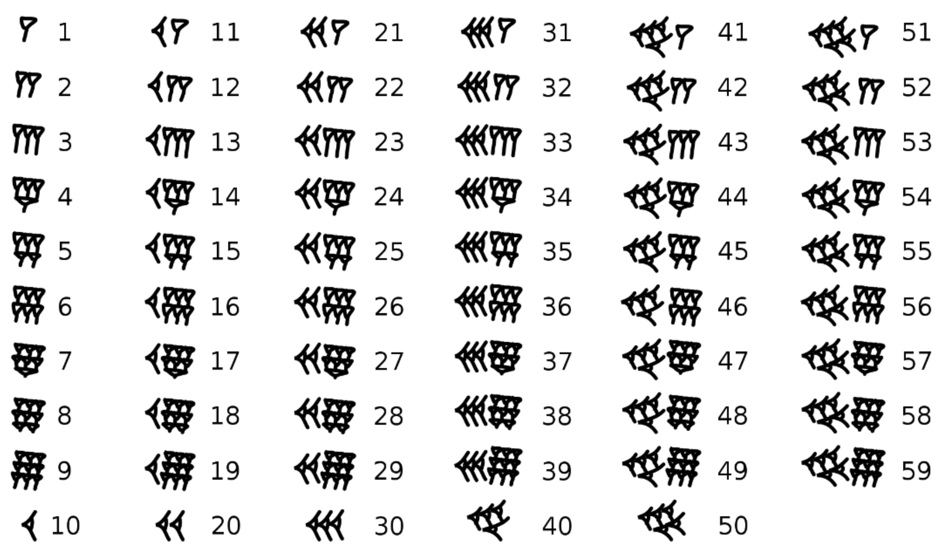
The most used prefix for metric system:

Atto a	10^{-18}	0.000 000 000 000 000 001
Femto f	10^{-15}	0.000 000 000 000 001
Pico p	10^{-12}	0.000 000 000 001
Nano n	10^{-9}	0.000 000 001
Micro µ	10^{-6}	0.000 001
Milli m	10^{-3}	0.001
Centi c	10^{-2}	0.01
Deci d	10^{-1}	0.1
	10^{0}	1
Deca da	10^{1}	10
Hecto h	10^{2}	100
Kilo k	10^{3}	1 000
Mega M	10^{6}	1 000 000
Giga G	10^{9}	1 000 000 000
Tera T	10^{12}	1 000 000 000 000

Numeral systems.

Over the long centuries of human history, many different numeral systems have appeared in different cultures. The oldest systems weren't a place-valued system, but sometimes use the position of the "digit" to show units. For example, the Babylonian system (form about 2000 BC) used only two symbols to write any number between 1 and 60:

 to count units and  to count tens.



By Josell7 - File:Babylonian_numerals.jpg, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=9862983

Number 62 was shown as  which means one time 60 and 2. The use of this sexagesimal (60-based) system is still visible, we have 60 minutes in one hour, 60 seconds in a minute, 360° for the full turn around.

Another very well-known numeral system is roman; it was used for thousands of years and in some cases is still used now as well. It's a "decimal", 10 based system, but the symbols (letters) are used in an unusual way.

Symbol	I	V	X	L	C	D	M
Value	1	5	10	50	100	500	1000

For example, 4 is one less than 5, so 4 can be written as IV. Same principle of subtractive notation is used for 9 -> IX, 40 and 90 -> XL and XC, 400 and 900 -> CD and CM

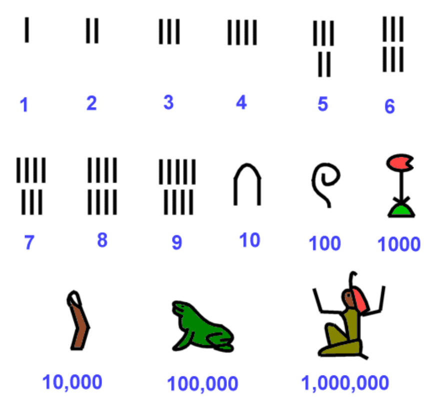
1	2	3	4	5	6	7	8	9	10
I	II	III	IV	V	VI	VII	VIII	IX	X

Some other examples:

- 29 = XX + IX = **XXXIX**.
- 347 = CCC + XL + VII = **CCCXLVI**.
- 789 = DCC + LXXX + IX = **DCCLXXXIX**.
- 2,421 = MM + CD + XX + I = **MMCDXXI**.

Any missing place (represented by a zero in the place-value equivalent) is omitted, as in Latin (and English) speech:

- 160 = C + LX = **CLX**
- 207 = CC + VII = **CCVII**
- 1,009 = M + IX = **MIX**
- 1,066 = M + LX + VI = **MLXVI**



Egyptian numeral system was decimal (10-based), similar to the one we use now, but also wasn't placed-valued and didn't use the position of the symbol in any way.
To write a number, you need to draw the symbol of units, tens, and so on as many times as there are units, tens, and so on. For example, the number 4622 should be written as (order in which all the symbols are written is not important).:

Can non-decimal place-value system be created? For example, with base 5?
Let see, how we can create this kind of system (we use our normal digits).

Num_{10}	1	2	3	4	5	6	7	8	9	10
Num_5	1	2	3	4	10	11	12	13	14	20

11	12	13	14	15	16	17	18	19	20
21	22	23	24	30	31	32	33	34	40

21	22	23	24	25	26	27	28	29	30
41	42	43	44	100	101	102	103	104	110

We only have 5 digits (0, 1, 2, 3, 4), and 4 first “natural” numbers in such system will be represented just with digits. Number 5 then should be shown as a 2-digit number, with fist digit 1 (place – value equal to 5^1) and 0 of “units”. Any number is now written in the form

$5^3 \cdot m + 5^2 \cdot n + 5^1 \cdot k + 5^0 \cdot p$

$33 = 25 + 5 + 3 = 5^2 \cdot 1 + 5^1 \cdot 1 + 3 \rightarrow 113_5$

$195 = 125 + 2 \cdot 25 + 20 = 5^3 + 5^2 \cdot 2 + 5^1 \cdot 4 + 0 \rightarrow 1240_5$

And vice versa, if we need transform the number form 5-base to decimal system:

$2312_5 \rightarrow 5^3 \cdot 2 + 5^2 \cdot 3 + 5^1 \cdot 1 + 5^0 \cdot 2 = 250 + 75 + 5 + 2 = 332$

Let’s try to introduce a new digit ├ for 10 and then create an 11 based system.

$11^2 = 121, \qquad 11^3 = 1331$

$890 = 847 + 33 + 10 = 121 \cdot 7 + 11 \cdot 3 + 10 = 11^2 \cdot 7 + 11^1 \cdot 3 + 11^0 \cdot 10 \rightarrow 73\vdash_{11}$

$4\vdash 2_{11} \rightarrow 11^2 \cdot 4 + 11^1 \cdot 10 + 11^0 \cdot 2 = 484 + 110 + 2 = 596$

There is another very important place-value system: binary system, base 2 system where only two digits exist; 0, and 1.

Num_{10}	1	2	3	4	5	6	7	8	9	10
Num_2	1	10	11	100	101	110	111	1000	1001	1010

In this system each number is represented as

$2^3 \cdot (0,1) + 2^2 \cdot (0,1) + 2^1 \cdot (0,1) + 2^0 \cdot (0,1)$

$11 = 8 + 2 + 1 = 2^3 \cdot 1 + 2^2 \cdot 0 + 2^1 \cdot 1 + 2^0 \cdot 1 \rightarrow 1011_2$

$75 = 64 + 8 + 2 + 1 = 2^6 + 2^3 + 2 + 1$

$= 2^6 \cdot 1 + 2^5 \cdot 0 + 2^4 \cdot 0 + 2^3 \cdot 1 + 2^2 \cdot 0 + 2^1 \cdot 1 + 2^0 \cdot 1 \rightarrow 1001011_2$

Exercises:

1. Represent a^{24} as an exponent with the base

$a.\ a^2;\quad b.\ a^3;\quad c.\ a^4;\quad d.\ a^6;\quad e.\ a^8;\quad f.\ a^{12}$

2. Compare the following exponents:

   $a.\ 2^{10}\ \ and\ \ 10^3;\qquad b.\ 10^{100}\ \ and\ 100^{10}$

   $c.\ 2^{300}\ \ and\ \ 200;\qquad d.\ 31^{16}\ \ and\ 17^{20};\qquad e.\ 4^{53}\ \ and\ 15^{45}$

3. Prove that
   $8^5 + 2^{11}$ is divisible by 17
   $9^7 - 3^{10}$ is divisible by 20
4. Write numbers 45 and 165 in binary system

5. Write the numbers, written in the binary system in decimal system:
   $a.\ 11011011;\qquad b. 10001101,\qquad c.\ 111111111$

6. Write the numbers 245 and 324 in 6-based place-value system. Remember, that in this system you will have only 0, 1, 2, 3. 4, and 5 as digits.

7. Write the numbers 234_6 and 403_6 written in the 6-based place-value system (small number 6 shows that the number is not in decimal, but in 6-based system) in decimal system.

8. How to arrange 127 dollar bills in seven wallets so that any amount from 1 to 127 dollars could be issued without opening the wallets?

9. Robert thought of a number not less than 1 and not more than 1000. Julia is allowed to ask only such questions to which Robert can answer “yes” or “no” (Robert always tells the truth). Can Julia determine the hidden number in 10 questions?

10. There is a bag of sugar, a scale and a weight of 1 g. Is it possible to measure 1 kg of sugar in 10 weights?