MATH 10
ASSIGNMENT 26: TOPOLOGY

MAY 7

Definition 1. Let X, Y be two subsets in $\mathbb{R}^3$ (or, more generally, any two metric spaces). We say that X, Y are topologically equivalent (homeomorphic) if there are continuous functions $f\colon X \to Y$, $g\colon Y \to X$ which are inverses of each other: $f \circ g = \mathrm{id}_Y$, $g \circ f = \mathrm{id}_X$ (where id_X is the identity map $X \to X$: $\mathrm{id}(x) = x$.)

1. Show that the open segment $(0, 1)$ is topologically equivalent to the halfline $(1, \infty)$.

2. Show that the open segment $(0, 1)$ is topologically equivalent to the real line.

3. Show that the circle $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ is topologically equivalent to the square (i.e. the boundary of the square, consisting of four segments, not including the interior).

4. Show that the plane with one point removed is topologically equivalent to the cylinder $S^1 \times \mathbb{R}$ (where S^1 is the circle.)

5. Show that the plane with a ray $[0, \infty)$ (positive part of the real line) removed is topologically equivalent to the halfplane.

6. Let us call a set X *connected* if any two points can be connected by a path: for any $x_0, x_1 \in X$ there exists a past γ in X, whose endpoints are x_0, x_1, i.e. a continuous map $\gamma\colon [0, 1] \to X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

Show that the real line is connected, but the set $\mathbb{R} - \{0\}$ is not connected. Deduce from this that $\mathbb{R}$ and $\mathbb{R} - \{0\}$ are not topologically equivalent.

7. Is the real line with one point removed topologically equivalent to the real line with two points removed?

8. Consider the sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ and the torus T. Are they topologically equivalent? can you argue why?