

**MATH 10**  
**ASSIGNMENT 10: OPEN AND CLOSED SETS**  
DEC 4, 2022

**Definition 1.** A metric space is a set  $X$  with a distance function: for any  $x, y \in X$  we have a real number  $d(x, y)$  such that

1.  $d(x, y) = d(y, x)$
2.  $d(x, y) \geq 0$  for any  $x, y$
3.  $d(x, y) = 0$  if and only if  $x = y$
4. Triangle inequality:  $d(x, y) + d(y, z) \geq d(x, z)$ .

Usual examples are  $\mathbb{R}, \mathbb{R}^2, \dots$ , but there are other examples as well.

Given a point  $x \in X$  and a positive real number  $\varepsilon$ , we define  $\varepsilon$ -neighborhood of  $x$  by

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

If  $S \subset X$ , denote by  $S'$  the complement of  $S$ . Then, for any  $x \in X$ , we can have one of three possibilities:

1. There is a neighborhood  $B_\varepsilon(x)$  which is completely inside  $S$  (in particular, this implies that  $x \in S$ ). Such points are called *interior points* of  $S$ ; set of interior points is denoted by  $\text{Int}(S)$ .
2. There is a neighborhood  $B_\varepsilon(x)$  which is completely inside  $S'$  (in particular, this implies that  $x \in S'$ ). Thus,  $x \in \text{Int}(S')$ .
3. Any neighborhood of  $x$  contains points from  $S$  and points from  $S'$  (in this case, we could have  $x \in S$  or  $x \in S'$ ). Set of such points is called the *boundary* of  $S$  and denoted  $\partial S$ .

**Definition 2.** A set  $S$  is called *open* if every point  $x \in S$  is an interior point:  $S = \text{Int}(S)$ .

A set  $S$  is called *closed* if  $\partial S \subset S$ .

HOMEWORK

1. Show that set  $\mathbb{R}^2$  with distance defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

is a metric space. (This distance is sometimes called *Manhattan* or taxicab distance — can you guess why?)

2. For each of the following subsets of  $\mathbb{R}$ , find its interior and boundary and determine if it is open, closed, or neither.
  - (a) Set  $\mathbb{N} = \{1, 2, 3, \dots\}$ .
  - (b) Interval  $[0, 1]$
  - (c) Open interval  $(0, 1)$
  - (d) Interval  $[0, 1)$ .
  - (e) Set of all rational numbers
  - (f) Set consisting of just two points  $\{0, 1\}$
  - \*(g) Set  $x^3 + 2x + 1 > 0$

Are there any subsets of  $\mathbb{R}$  which are both open and closed?

3. Show that a set  $S$  is open if and only if its complement  $S'$  is closed.

\*4. For a set  $S$ , let  $\bar{S} = S \cup \partial S = \{x \mid \text{In any neighborhood of } x, \text{ there are elements of } S\}$ . Prove that  $\bar{S}$  is closed. (It is called the closure of  $S$ .)

5. Show that union and intersection of two open sets is open. Is it true if we replace two sets by any collection of open sets?

Same question about closed sets.

\*6. Consider the set  $P$  of all polynomials in variable  $x$  with real coefficients. Define the **norm** of a polynomial by

$$\|p\| = 2^{-n} \quad \text{if } p(x) = x^n(a_0 + a_1x + \dots), \quad a_0 \neq 0$$

For example, for any polynomial that has non-zero constant term,  $\|p\| = 1$ ; for a polynomial which has zero as a root of multiplicity  $n$ ,  $\|p\| = 2^{-n}$ .

Define now distance between two polynomials by

$$d(f, g) = \|f - g\|.$$

For example, distance between  $1+x+x^2+x^7$  and  $1+x+x^3$  is  $2^{-2}$ , since their difference is  $x^2-x^3+x^7$ .

Prove that this satisfies the properties of distance; moreover, in this case we have a stronger condition: given 3 polynomials  $f, g, h$ , we have

$$d(f, h) \leq \max(d(f, g), d(g, h))$$