MATH 10
ASSIGNMENT 7: SYSTEMS OF LINEAR EQUATIONS
NOVEMBER 6, 2022

Systems of linear equations

A number of practical problems give a number of linear equations which have to be simultaneously solved. For example, suppose sudents A and B went to the supermarket. Student A bought a bag of cookies and two cans of soda, paying five dollars. Student B bought two bags of cookies and a can of soda and paid seven dollars. How much does each item cost?

It is easy to see that this problem can be phrased in terms of a *system of linear equations*. Let c denote the price of the bag of cookies and s the price of a can of soda. Then

$$c + 2s = 5$$
$$2c + s = 7$$

An intuitive method to solve this problem is that of substitution. First we solve the first equation for c:

$$c = 5 - 2s. \tag{1}$$

Then we substitute this result in the first equation and solve for s:

$$2(5 - 2s) + s = 7 \Rightarrow s = 1$$

Finally, we go backwards, substituting $s = 1$ back in equation (1): $c = 5 - 2(1) = 3$.

We will now start the study of general systems of linear equations, with m equations and n variables. They will look something like this:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

To such a system of equations we associate the *matrix of coefficients*

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

and the *vector of constants*

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

We can combine these two into the *augmented matrix*,

$$A|\mathbf{b} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & | & b_m \end{bmatrix},$$

For example, if the systems of linear equations is

$$2x_1 + x_2 + 3x_3 = 2$$
$$4x_1 - 7x_2 + 5x_3 = 1,$$

then we get

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -7 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A|\mathbf{b} = \begin{bmatrix} 2 & 1 & 3 & | & 2 \\ 4 & -7 & 5 & | & 1 \end{bmatrix}.$$

Elementary row operations

The main idea of the method we will use to solve an arbitrary system of linear equations is to transform it to a simpler form. The transformation should not change the set of solutions. To do this, we will use the following elementary operations:

- Exchange two equations (= two rows of the augmented matrix)
- Multiply both sides of an equation (= one row of augmented matrix) by a non-zero number
- Add to one equation a multiple of another (= add to one row of the augmented matrix a multiple of another)

Applying these operations to bring your matrix to a simpler form is called *row reduction*, or *Gaussian elimination*

Simple example

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 5 \\ x_1 - x_2 &= -1 \\ -x_1 + x_2 + x_3 &= 5 \end{aligned}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -2 & 2 & | & 5 \\ 1 & -1 & 0 & | & -1 \\ -1 & 1 & 1 & | & 5 \end{bmatrix}$$

Using row operations, we can bring it to the form

$$\begin{bmatrix} 1 & -2 & 2 & | & 5 \\ 0 & 1 & -2 & | & -6 \\ 0 & 0 & 1 & | & 4 \end{bmatrix}$$

so the solution is

$$\begin{aligned} x_3 &= 4 \\ x_2 &= -6 + 2x_3 = 2 \\ x_1 &= 5 - 2x_3 + 2x_2 = 1 \end{aligned}$$

Row echelon form

In general, using row operations, every system can be brought to a form where each row begins with some number of zeroes, and each next row has more zeroes than the previous one:

$$\begin{bmatrix} X & * & * & * & * & * & * & | & * \\ 0 & 0 & 0 & X & * & * & * & | & * \\ 0 & 0 & 0 & 0 & X & * & * & | & * \end{bmatrix}$$

(here X's stand for non-zero entries).

To solve such a system, we do the following:

- Variables corresponding to columns with X's in them are called pivot variables; the remaining ones are called free variables.
- Values for free variables can be chosen arbitrarily. Values for pivot variables are then uniquely determined from the equations.

For example, in the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_2 + 3x_3 &= 6 \end{aligned}$$

varaibles x_1, x_2 are pivot, and variable x_3 is free, so we can solve it by letting $x_3 = t$, and then

$$\begin{aligned} x_2 &= 6 - 3x_3 = 6 - 3t \\ x_1 &= 5 - x_2 - x_3 = -1 + 2t \end{aligned}$$

Make sure you understand what the meaning here is: for any real number t,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1+2t \\ 6-3t \\ t \end{bmatrix}$$

is a solution.

Homework

1. Solve the following system of equations

$$\begin{aligned} w + x + y + z &= 6 \\ w + y + z &= 4 \\ w + y &= 2 \end{aligned}$$

2. Solve the system of equations with the following matrix

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{array}\right]$$

3. Solve the following system of equations

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 3 \\ -x_1 + x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + 8x_3 &= 4 \end{aligned}$$

4. Consider the system of equations

$$\begin{aligned} 3x - y + 2z &= b_1 \\ 2x + y + z &= b_2 \\ x - 7y + 2z &= b_3 \end{aligned}$$

(a) If $b_1 = b_2 = b_3 = 0$, find all solutions

(b) For which triples b_1, b_2, b_3 does it have a solution?

5. Consider a system of 4 equations in 5 variables.

(a) Show that if the right-hand side is zero, then this system must have a non-zero solution.

(b) Is it true if the right-hand side is non-zero?

6. Find a vector of unit length perpendicular to the plane determined by the points $P(1, 2, 3)$, $Q(-1, 3, 2)$, $R(3, -1, 2)$, and find the area of the triangle PQR.