

**MATH 10**  
**ASSIGNMENT 6: CROSS-PRODUCT**  
 OCT 30, 2022

SIGNED AREA: REVIEW

Recall that we had defined “wedge product” of two vectors in the plane by

$$(1) \quad \mathbf{v} \wedge \mathbf{w} = x_1y_2 - y_1x_2 \in \mathbb{R}$$

One can think of  $\mathbf{v} \wedge \mathbf{w}$  as “signed area”:

$$\mathbf{v} \wedge \mathbf{w} = \begin{cases} S_{ABCD}, & \text{if rotation from } \mathbf{v} \text{ to } \mathbf{w} \text{ is counterclockwise} \\ -S_{ABCD}, & \text{if rotation from } \mathbf{v} \text{ to } \mathbf{w} \text{ is clockwise} \end{cases}$$

The wedge product (and thus, the signed area) is in many ways easier than the usual area. Namely, we have:

1. It is linear:  $(\mathbf{v}_1 + \mathbf{v}_2) \wedge \mathbf{w} = \mathbf{v}_1 \wedge \mathbf{w} + \mathbf{v}_2 \wedge \mathbf{w}$
2. It is anti-symmetric:  $\mathbf{v} \wedge \mathbf{w} = -\mathbf{w} \wedge \mathbf{v}$

CROSS-PRODUCT

If  $\mathbf{v}, \mathbf{w}$  are two vectors in  $\mathbb{R}^3$ , then we can define a different kind of product, called the cross-product, which is a **vector** in  $\mathbb{R}^3$ , defined by

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1z_2 - z_1y_2 \\ z_1x_2 - x_1z_2 \\ x_1y_2 - y_1x_2 \end{bmatrix}$$

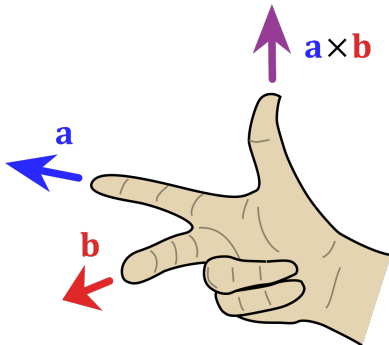
For example, if  $\mathbf{v}, \mathbf{w}$  are vectors in the  $xy$  plane, then  $\mathbf{v} \times \mathbf{w}$  is a vector along the direction of the  $z$ -axis; moreover, in this case

$$\mathbf{v} \times \mathbf{w} = (\mathbf{v} \wedge \mathbf{w})\mathbf{j}$$

where  $\mathbf{j}$  is the unit vector in the positive direction of  $z$ -axis.

The cross-product has several important properties. Some of them are proved below, others are left without a proof.

1. It is linear in  $\mathbf{v}, \mathbf{w}$ :  $(\mathbf{v}' + \mathbf{v}'') \times \mathbf{w} = \mathbf{v}' \times \mathbf{w} + \mathbf{v}'' \times \mathbf{w}$ , and similarly for  $\mathbf{w}$
2. It is anti-symmetric:  $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$
3.  $|\mathbf{v} \times \mathbf{w}| = \text{area of the parallelogram with sides } \mathbf{v}, \mathbf{w}$
4.  $\mathbf{v} \times \mathbf{w}$  is perpendicular to the plane containing  $\mathbf{v}, \mathbf{w}$
5. The direction of  $\mathbf{v} \times \mathbf{w}$  is determined by so-called right hand rule:



Thus, if  $\mathbf{v}$  is along positive direction of  $x$  axis, and  $\mathbf{w}$  is in the positive direction of  $y$ -axis, then  $\mathbf{v} \times \mathbf{w}$  will be in the positive direction of the  $z$ -axis.

1. Check that if  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are unit vectors along positive directions of  $x, y, z$  axes respectively, then

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

and similar for the cyclic permutations of these three vectors:  $\mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$

2. (a) Use cross-product to construct a vector perpendicular to both of the vectors below:

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- (b) Write the equation of the plane through points  $(0, 0, 0), (1, 0, 2), (1, 1, 1)$ .
3. Show that if all vertices of a triangle in the plane have integer coordinates, then its area  $A$  is a half-integer (i.e.,  $2A \in \mathbb{Z}$ ). Is the same true for any polygon?
4. For three vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  in  $\mathbb{R}^3$ , define the triple product  $T(\mathbf{u}, \mathbf{v}, \mathbf{w})$  by the formula

$$T(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$$

(note that it is a number, not a vector). The notation  $T$  is not standard.

- (a) Write an explicit formula the triple product in terms of  $x, y,$  and  $z$  components of  $\mathbf{u}, \mathbf{v}, \mathbf{w}$ .
- (b) Check that the triple product is linear in each of the three vectors and is anti-symmetric:

$$T(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -T(\mathbf{v}, \mathbf{u}, \mathbf{w})$$

and similarly for any other transposition (interchange of any two of the three vectors).

- (c) Deduce from part (b) that  $T(\mathbf{u}, \mathbf{v}, \mathbf{w}) = T(\mathbf{w}, \mathbf{u}, \mathbf{v})$ . [Hint: one can get triple  $\mathbf{w}, \mathbf{u}, \mathbf{v}$  from  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  by two transpositions ]
- (d) Deduce from part (b) that  $T(\mathbf{v}, \mathbf{v}, \mathbf{w}) = 0$  and thus  $\mathbf{v} \times \mathbf{w}$  is perpendicular to  $\mathbf{v}$ .
- (e) Show that for a parallelepiped  $P$  with edges given by vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$ , its volume is given by

$$V_P = |T(\mathbf{u}, \mathbf{v}, \mathbf{w})|$$

5. What is the volume of a tetrahedron with vertices  $A = (0, 0, 0), B = (x_1, y_1, z_1), C = (x_2, y_2, z_2), D = (x_3, y_3, z_3)$ ?