

MATH 10
ASSIGNMENT 3: DOT PRODUCTS
OCTOBER 2, 2022

DOT PRODUCT

By the Pythagorean theorem, for a vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, its length is given by $\sqrt{x^2 + y^2 + z^2}$. It is common to denote the length of a vector \mathbf{v} by $|\mathbf{v}|$:

$$|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$$

A convenient tool for computing lengths is the notion of the *dot product*. The dot product of two vectors is a number (not a vector!) defined by

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

The dot product has the following properties:

1. It is symmetric: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
2. It is linear as function of \mathbf{v} , \mathbf{w} :

$$(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w} = \mathbf{v}_1 \cdot \mathbf{w} + \mathbf{v}_2 \cdot \mathbf{w}$$

$$(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w})$$

3. $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$, or, equivalently, $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$
4. Vectors \mathbf{v} , \mathbf{w} are perpendicular iff $\mathbf{v} \cdot \mathbf{w} = 0$.

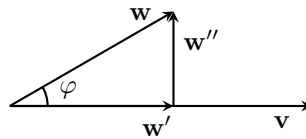
The first three properties are immediate from the definition. The last one follows from the Pythagorean theorem: if $\mathbf{v} \perp \mathbf{w}$, then by Pythagorean theorem, $|\mathbf{v}|^2 + |\mathbf{w}|^2 = |\mathbf{v} - \mathbf{w}|^2 = (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} - 2\mathbf{v} \cdot \mathbf{w}$.

From these properties one easily gets the following important result:

Theorem.

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}' = |\mathbf{v}| \cdot |\mathbf{w}| \cos \varphi$$

where φ is the angle between vectors \mathbf{v} , \mathbf{w} , and \mathbf{w}' is projection of \mathbf{w} onto \mathbf{v} . It can be defined by saying that $\mathbf{w} = \mathbf{w}' + \mathbf{w}''$, and vector \mathbf{w}' is a multiple of \mathbf{v} , \mathbf{w}'' is perpendicular to \mathbf{v} :

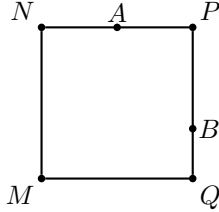


This theorem is commonly used to find the angle between two vectors:

$$\cos \varphi = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}| \cdot |\mathbf{w}|}$$

HOMEWORK

1. Prove that the triangle with vertices at $A(3, 0)$, $B(1, 5)$, and $C(2, 1)$ is obtuse. Find the cosine of the obtuse angle.
2. Prove the law of cosines: in a triangle $\triangle ABC$, with sides $AB = c$, $AC = b$, $BC = a$, one has $c^2 = a^2 + b^2 - 2ab \cos \angle C$. [Hint: $c^2 = \vec{AB} \cdot \vec{AB}$, and $\vec{AB} = \vec{CB} - \vec{CA}$.]
3. (a) Let \mathbf{v} , \mathbf{w} be two vectors in the plane. Show that if $|\mathbf{v}| = |\mathbf{w}|$, then $(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = 0$.
 (b) Use the previous part to show that the two diagonals of a rhombus are perpendicular.
4. On the sides of a square $MNPQ$, with side 1, the points A and B are taken so that $A \in NP$, $NA = \frac{1}{2}$, $B \in PQ$, and $QB = \frac{1}{3}$. Prove that $\angle AMB = 45^\circ$.



5. Use dot product to find the angle between two diagonals of unit cube. (You will need to first use write each diagonal in coordinates, as a vector.)
6. A billiard ball traveling with velocity \vec{v} hits another ball which was at rest. After the collision, balls move with velocities \vec{v}_1 , \vec{v}_2 . Prove that $\vec{v}_1 \perp \vec{v}_2$, using the following conservation laws (m is the mass of each ball which is supposed to be the same)

Momentum conservation: $m\vec{v} = m\vec{v}_1 + m\vec{v}_2$

Energy conservation: $\frac{m|\vec{v}|^2}{2} = \frac{m|\vec{v}_1|^2}{2} + \frac{m|\vec{v}_2|^2}{2}$

7. Consider the plane given by equation $ax + by + cz = d$.
 (a) Let $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ be two points on this plane. Prove that then

$$a(x_1 - x_2) + b(y_1 - y_2) + c(z_1 - z_2) = 0.$$

- (b) Prove that $\vec{P_1P_2}$ is perpendicular to vector $\mathbf{v} = (a, b, c)$.
 (In such a situation, we say the plane is perpendicular to \mathbf{v} .)