PROBABILITY

APRIL 16, 2023

Expected values

If a quantity X depends on result of some random experiment (e.g. tossing a coin), taking values $a_1, a_2, \dots$ with probabilities $p_1, p_2, \dots$ respectively, the **expected value** of X is defined as

$$E(X) = p_1 a_1 + p_2 a_2 + \dots$$

Informally, if you repeat the experiment N times and denote by $X_1, X_2, \dots$ the values of X you received in these expereiments, then for large N, we have

$$\frac{X_1 + X_2 + \dots}{N} \approx E(X)$$

Important property of the expected value is that if you have two random variables X, Y (both depending on result of same random experiment), then

$$E(X + Y) = E(X) + E(Y)$$

regardless of whether X, Y are independent of each other or not. Note that it only works for the sum; for example, $E(XY)$ in general is not equal to $E(X)E(Y)$.

1. On average, how many times must a die be thrown until one gets a 6?

2. Coupons in cereal boxes are numbered from 1 to 5, and a set of one of each is required for a prize.
  (a) If you already have 2 different kinds of coupon, how many more boxes on average do you need to buy before you get a 3rd one?
  (b) With one coupon per box, how many boxes on the average are required to make a complete set?

3. Sophie is tossing a fair coin.
  (a) On average, how many times does she need to toss a coin before she sees two heads in a row?
  (b) On average, how many times does she need to toss a coin before she sees sequence HTHT? sequence THTT?

4. Anne and Bob toss a fair coin. Anne tosses it 31 times, Bob, 30 times.
What is the probability that Anne got more heads than Bob?
[Hint: denote the probability that after 30 tosses Anne has more heads than Bob by p. Express your answer in terms of p.]

5. [Crazy old lady problem]
A plane has 100 seats; all tickets for thsi flight were sold, so there are exaclty 100 passengers waiting to board, each with a ticket. The first in line to board is a creazy old lady. She enters the plane and takes a random seat.
After that, each passenger enters the plane and takes his/her seat if it is available; otherwise, they take a random available seat.
What is the probability that the last passenger will be seated in a wrong seat?

6. [Banach's matchboxes] Polish mathematician Stephan Banach used to carry a matchbox in each of his coat pockets (left and right). Every time he wanted to light up his pipe, he would get one of the mathboxes at random and use it.
Initially each of matchboxes contained n matches. However, one day he reached for a matchbox and it was empty.
  (a) What is the probabilyt that at this moment, the other matchbox had k matches?
  (b) What is the average (i.e. expected value) numebr of matches in the remaining matchbox?

7. Sam has a one meter long piece of rope. He cuts it at a random place.
What is the average length of the shorter piece?

8. Alice has a loop of rope, one meter long. She holds it in one place; Boris takes scissors and (with his eyes closed) cut the rope at two random places. Alice takes the piece she was holding.

(a) What is the probability that Alice got the shorter piece?
(b) What is the average length of the piece of rope that Alice gets?

9. In a bag there are n pieces of rope. Shurik ties all loose ends of rope to each other at random (in particular, it coudl happen that he ties together two ends of the same piece.)
(a) What is the probability that as a result, we will get a single loop?
(b) How many loops, on average, will we get?