ADVANCED MATH PROBLEM SOLVING CLUB
ASSIGNMENT 2: GRAPHS

SEPT 25, 2022

Graph theory

In mathematics graph is a set V of objects (called *vertices, nodes*, or *points*) together with a set E of *edges, arcs*, or *lines*. Each edge connects two vertices (it is also allowed to have an edge which connects a vertex to itself — a loop edge). We will always assume that the graphs are finite: finitely many vertices and finitely many edges.

Very often graphs are used to represent complicated data sets in an economic way. They can be very useful in solving mathematical problems. However, graphs are also very interesting on their own. The branch of mathematics studying graphs with various additional structures is called *graph theory*.

There is a lot of terminology related to graphs, see: `https://en.wikipedia.org/wiki/Graph`. Here we will list just some of it:

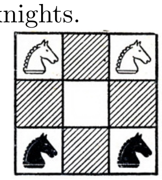
- For a vertex v in a graph, its *degree* is the number of edges meeting at that vertex. [Loop edges count twice.]
- A graph is called *connected* if you can connect any two vertices by a path along the edges of the graph. Otherwise, it is called a disconnected graph.
- A *tree* is a connected graph without loops; in such a graph, any two vertices are connected by exactly one path.

Simple problems

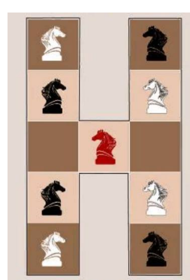
1. Is there a graph with 13 vertices, each connected to exactly 5 other vertices?

2. Is it possible to draw 9 segments on the plane so that each of them intersects with exactly 3 other segments?

3. In the picture below, find the shortest sequence of moves which would exchange the positions of the two black knights with the two white knights.

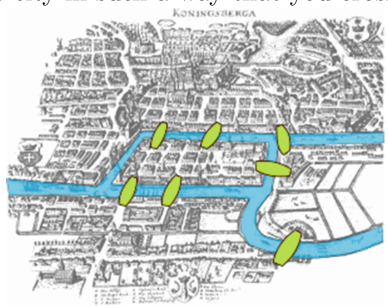


4. In the picture below, find the shortest sequence of moves which would exchange the positions of the black knights with the white knights, leaving the red knight (the one in the center) in its original position.



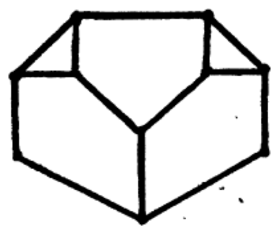
5. Bridges of Königsberg. The picture below shows the map of a city of Königsberg (modern day Kaliningrad) as it was at the time of Leonhard Euler. The map shows the river and seven bridges that existed at that time.

Is it possible to walk in the city in such a way that you cross each bridge exactly once?



6. A graph is called *Eulerian* if it can be traced on paper (without lifting the pencil) so that each edge is traced exactly once. Show that a connected graph is Eulerian if and only if it has at most two odd vertices (odd vertex is the vertex which belongs to odd number of edges).

7. The picture below shows a map of a park; black lines are hedges. In each hedge there is one gate. Can one have a walk in the park (and outside the park) going through every gate only once?



8. Is it possible to make a skeleton of the cube $10 \times 10 \times 10$ from the wire of the length 120 without cutting the wire?

9. In a country X there are 101 cities. Some of the cities are connected by roads so that there is only one path between any two cities. How many roads are there in the country X?

10. Show that the number of vertices V of a finite tree exceeds the number of edges E by one: $V = E + 1$.

11. King Solomon had 5 sons and no daughters. Out of all his descendants, 100 had exactly 3 sons while others died not leaving any children. How many descendants did Solomon have?

Harder problems

For those who know how to do all the previous ones.

1. In a country Y there are 7 lakes connected by 10 channels so that one can boat from any lake to any other. How many islands are in the country Y? [All islands are formed by lakes and channels; there are no islands inside the lakes.]

2. Someone marked 20 points inside a square and connected points with each other and with vertices of the square. As a result the square is divided into triangles. How many triangles are there?

3. Suppose that we want to color the whole plane so that any two points at distance 1 from each other would have different colors.

(a) Show that we would need at least 3 colors.

(b) Show that we would need at least 4 colors.

(This is known as the chromatic problem for the plane, or Hadwiger–Nelson problem. It is known that it can be done using 7 colors; it was recently – about 3 years ago! – proved that you need at least 5. It is an open problem to find the smallest number of colors.)