MATH 9
ASSIGNMENT 20: EQUIVALENCE RELATIONS
MARCH 20, 2022

Equivalence relations

A relation $\sim$ on a set A is called:

- **reflexive** if for any $a \in A$, we have $a \sim a$
- **symmetric** if for any $a, b \in A$, we have $a \sim b \implies b \sim a$
- **transitive** if for any $a, b, c \in A$, we have $a \sim b$ and $b \sim c \implies a \sim c$

A relation is called *an equivalence relation* if it is reflexive, symmetric, and transitive.

Given an equivalence relation on A, we can define, for every $a \in A$, its *equivalence class* $[a]$ as the following subset of A:

$$[a] = \{x \in A \mid x \sim a\}$$

Homework

1. For each of the following relations, check whether it is an equivalence relation.
(a) On the set of all lines in the plane: relation of being parallel
(b) On the set of all lines in the plane: relation of being perpendicular
(c) On $\mathbb{R}$: relation given by $x \sim y$ if $x + y \in \mathbb{Z}$
(d) On $\mathbb{R}$: relation given by $x \sim y$ if $x - y \in \mathbb{Z}$
(e) On $\mathbb{R}$: relation given by $x \sim y$ if $x > y$
(f) On $\mathbb{R} - \{0\}$: relation given by $x \sim y$ if $xy > 0$

2. Let $\sim$ be an equivalence relation on A.
(a) Prove that if $a \sim b$, then $[a] = [b]$: for any x,

$$x \in [a] \iff x \in [b]$$

(b) Prove that if $a \not\sim b$, then $[a] \cap [b] = \varnothing$.

3. Let $f \colon A \to B$ be a function. Define a relation on A by $a \sim b$ if $f(a) = f(b)$. Prove that it is an equivalence relation.

4. Fix a positive integer number n and define relation $\equiv$ on $\mathbb{Z}$ by

$$a \equiv b \text{ if } a - b \text{ is a multiple of } n$$

(a) Prove that it is an equivalence relation.
(b) Describe equivalence class $[0]$.
(c) Prove that equivalence class of $[a + b]$ only depends on equivalence classes of a, b, that is, if $[a] = [a']$, $[b] = [b']$, then $[a + b] = [a' + b']$.

5. Define a relation $\sim$ on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ by $(x_1, x_2) \sim (y_1, y_2)$ if $x_1 + x_2 = y_1 + y_2$. Prove that it is an equivalence relation and describe the equivalence class of $(1, 2)$.

6. Consider the set $A = \{(a, b) \mid a, b \in \mathbb{Z}, b \neq 0\}$. Define a relation by

$$(a_1, b_1) \sim (a_2, b_2) \text{ if } a_1 b_2 = a_2 b_1$$

Prove that it is an equivalence relation.

Hint:

$$a_1 b_2 = a_2 b_1 \iff \tfrac{a_1}{b_1} = \tfrac{a_2}{b_2}$$

***7.** Consider the set $A = \mathbb{R}^2 - \{(0, 0)\}$ (coordinate plane with the origin removed). Define a relation $\sim$ on A by

$$(x_1, x_2) \sim (y_1, y_2) \text{ if there exists } t > 0 \text{ such that } x_1 = t y_1, x_2 = t y_2$$

(a) Prove that it is an equivalnce relation.
(b) Describe the equivalence class of $(1, 2)$.