MATH 9
ASSIGNMENT 18: COMPARING INFINITE SETS
FEB 27, 2022

Topics discussed today

Today we discussed how one compares infinite sets. Namely, we adopted the following definition:two (infinite) sets A, B *have the same cardinality* if there exists a bijection $f\colon A \to B$. In this case, we also write $|A| = |B|$ (note that both $|A|$, $|B|$ are not numbers but "infinities").

In particular, the smallest infinite set is $\mathbb{N}$, the set of all positive integer numbers: $\mathbb{N} = \{1, 2, \dots\}$. We say that an infinite set A is *countable* if there is a bijection between A and $\mathbb{N}$.

We have proved in class that any subset of $\mathbb{N}$ is either finite or countable; we also proved that $\mathbb{Z}$, and the set of pairs of positive integers $\mathbb{N} \times \mathbb{N} = \{(a, b) | A \in \mathbb{N}, b \in \mathbb{N}\}$ are countable.

Homework

Problems 1—4 are about *Hotel Infinity*, a (fictional) hotel with infinitely many rooms, numbered 1, 2, 3,.... In these questions, we assume that each hotel room is single occupancy: only one guest can stay there at any time.

Each of these problems is also reformulated as a question about sets.

1. (a) At some moment, Hotel Infinity is full: all rooms are occupied. Yet, when 2 more guests arrive, the hotel manager says he can give rooms to them, by moving some of the current guests around. Can you show how?
(b) Construct a bijection between sets $\{-1, 0, 1, 2, \dots\}$ and $\mathbb{N}$.

2. (a) At some moment, Hotel Infinity is full: all rooms are occupied. Still, the management decides to close half of the rooms — all rooms with odd numbers — for renovation. They claim they can house all their guests in the remaining rooms. Can you show how?
(b) Construct a bijection between the set of all even positive integers $\{2, 4, 6, \dots\}$ and $\mathbb{N}$.

3. (a) Next to Hotel infinity, a competitor has built Hotel Infinity 2, with infinitely many rooms numbered by all integers: $\dots, -2, -1, 0, 1, 2, \dots$. Yet, the management of original Hotel Infinity claims that their hotel is no smaller than the competition: they could house all the guests of Hotel Infinity 2 in Hotel Infinity. Could you show how?
(b) Construct a bijection between the set of all integer numbers $\{\dots, -2, -1, 0, 1, 2, \dots\}$ and $\mathbb{N}$.

***4.** (a) Next to Hotel infinity, a new competitor has built Hotel Infinity 3, with infinitely many rooms numbered by all positive rational numbers: there are rooms with numbers 1, 2, 2/5, 1/3, 1137/295, ... Yet, the management of original Hotel Infinity claims that their hotel is no smaller than the competition: they could house all the guests of Hotel Infinity 3 in Hotel Infinity. Could you show how?
(b) Construct a bijection between the set of all positive rational numbers and $\mathbb{N}$.

5. Prove that if A is finite and B countable, then $A \cup B$ is also countable. [Hint: look at problem 1.]

***6.** Let W be the set of all "words" that can be written using the alphabet consisitng of 26 lowercase English letters; by a "word", we mean any (finite) sequence of letters, even if it makes no sense – for example, *abababaaaaa*. Prove that W is countable. [Hint: for any n, there are only finitely many words of length n.]