MATH 9
ASSIGNMENT 17: ONTO AND ONE-TO-ONE FUNCTIONS. BIJECTIONS
FEBRUARY 13, 2022

Let $f\colon A \to B$ be a function.

Let $y \in B$ and consider the equation

$$f(x) = y$$

If such an equation always (i.e., for any $y \in B$) has **at least** one solution, we say that function f is **onto**, or **surjective**.

If such an equation always (i.e., for any $y \in B$) has **at most** one solution, we say that function f is **one-to-one**, or **injective**.

If such an equation always (i.e., for any $y \in B$) has a **exactly** one solution, we say that function f is **bijective**. Sometimes such functions are also referred to as or **one-to-one correspondences**.

If f is bijective, then it can be inverted: there is a function $g\colon B \to A$ such that $g(f(x)) = x$, $f(g(y)) = y$, namely,

$$g(y) = \text{ solution of equation } f(x) = y$$

Such a function is called **inverse of** f and denoted $g = f^{-1}$. For example, if $f(x) = 2x + 1$, then to find $f^{-1}(y)$, we need to solve $2x + 1 = y$, which gives $x = \frac{y-1}{2}$, so $f^{-1}(y) = \frac{y-1}{2}$. [In fact, it can be shown that conversely, if f has an inverse, then f must be a bijection.]

Bijections can be thought of as ways of identifying two different sets. In particular, if there exists a bijection f between two finite sets A, B, then they have the same number of elements: $|A| = |B|$.

1. Show that $f\colon A \to B$ is **not** injective if and only if there exist $x_1, x_2 \in A$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

2. For each of the following functions, determine whether it is injective? surjective? bijective?
   (a) $f_1\colon \mathbb{R} \to \mathbb{R}$, $f_1(x) = x^3 + 1$.
   (b) $f_2\colon \mathbb{R} \to \mathbb{R}$, $f_2(x) = x^2 - 2x$.
   (c) $f_3\colon \mathbb{R}_+ \to \mathbb{R}_+$, $f_3(x) = \sqrt{x}$ (here $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$).

3. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = 2n$. Is this function injective? surjective?

***4.** Let $f\colon A \to B$, $g\colon B \to C$ be bijections. Prove that the composition $g \circ f\colon A \to C$, defined by $g \circ f(x) = g(f(x))$, is also a bijection, and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

5. Construct bijections between the following sets:
   (a) (subsets of the set $\{1, \ldots, n\}$) $\leftrightarrow$ (sequences of zeros and ones of length n)
   (b) (5-element subsets of $\{1, \ldots, 15\}$)$\leftrightarrow$ (10-element subsets of $\{1, \ldots, 15\}$)
   (c) (set of all ways to put 10 books on two shelves (order on each shelf matters))$\leftrightarrow$ (set of all ways of writing numbers 1, 2, …, 11 in some order)
   [Hint: use numbers 1…10 for books and 11 to indicate where one shelf ends and the other begins.]
   (d) (all integer numbers)$\leftrightarrow$ (all even integer numbers)
   (e) (all positive integer numbers)$\leftrightarrow$ (all integer numbers)
   (f) (interval (0,1))$\leftrightarrow$ (interval (0,5))
   (g) (interval (0,1))$\leftrightarrow$ (halfline $(1, \infty)$) [Hint: try $1/x$.]
   (h) (interval (0,1))$\leftrightarrow$ (halfline $(0, \infty)$)
   (i) (all positive integer numbers)$\leftrightarrow$ (all integer numbers)

6. Let A be a finite set, with 10 elements. How many bijections $f\colon A \to A$ are there? What if A has n elements?