# MATH 9
## ASSIGNMENT 16: FUNCTIONS, IMAGES, AND PREIMAGES.
FEB 6, 2022

### Reminder: sets

Here is a brief reminder of set theory notations:

- $\varnothing$: empty set
- $x \in A$ (reads $x$ is in $A$): $x$ is an element of set $A$
- $\{x \mid \text{some condition on } x\}$: set-builder notation: set of all $x$ satisfying some condition
- $A \subset B$: set $A$ is a subset of $B$. Note that this includes possibility that $A = B$.
- $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$: union of two sets
- $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$: intersection of two sets
- $\overline{A} = \{x \mid x \notin A\}$: complement of $A$ (only makes sense if we specify what kind of values $x$ is allowed to take).

### Functions

A function $f\colon A \to B$ is some rule assigning to **every** element $x \in A$ an element $f(x) \in B$. Set $A$ is called **domain** of $f$ and set $B$, the **codomain**.

Note that we do not require that every element $y \in B$ appears as a value of our function; the set of values $\{y \in B \mid y = f(x) \text{ for some } x \in A\}$ is denoted $f(A)$ and called **range** of $f$. More generally, for every subset $X \subset A$ we denote

$$f(X) = \{y \in B \mid y = f(x) \text{ for some } x \in X\}$$

and call it the **image** of $X$.

**Preimage**: for a function $f\colon A \to B$, and any subset $C \subset B$, we denote

$$f^{-1}(C) = \{x \in A \mid f(x) \in C\},$$

and call it the **preimage** of $C$.

In particular, we can take $C = \{y\}$ to be a single point in $B$; in this case, $f^{-1}(\{y\})$ is the set of all solutions of the equation

$$f(x) = y.$$

**1.** Find the following images
   (a) $f([0,1])$, where $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 3x + 1$.
   (b) $f([-1,2])$, where $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = |x| - 1$.

**2.** For each of the following functions, find $f([0,3])$, $f^{-1}([0,3])$.
   (a) $f_1\colon \mathbb{R} \to \mathbb{R}$, $f_1(x) = x^3 + 1$.
   (b) $f_2\colon \mathbb{R} \to \mathbb{R}$, $f_2(x) = x^2 - 2x$.
   (c) $f_3\colon \mathbb{R}_+ \to \mathbb{R}_+$, $f_3(x) = \sqrt{x}$ (here $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$).

**3.** For two sets $A, B$, define their difference $A - B$ by $A - B = \{x \mid x \in A, x \notin B\} = A \cap \overline{B}$ (some books also use notation $A \setminus B$ instead of $A - B$).
   (a) Prove that $A - (B \cup C) = (A - B) - C$, but that in general, $A - (B - C) \neq (A \cup C) - B$ (draw Venn diagrams to help you).
   (b) Define symmetric difference $A \triangle B = (A - B) \cup (B - A)$. Prove that this operation is commutative and associative: $A \triangle B = B \triangle A$, $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

**4.** Let $f\colon A \to B$. Prove that for any two subsets $Y_1, Y_2 \subset B$, we have $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$. [Hint: $x \in f^{-1}(Y_1 \cap Y_2) \iff f(x) \in Y_1 \cap Y_2$. Now, we need to show that this condition is equivalent to $x \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$]

**5.** Let $f\colon A \to B$, and $X_1, X_2$ are subsets in $A$.
   (a) Prove that $f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$.
   (b) Show that it could happen that $f(X_1 \cap X_2) \neq f(X_1) \cap f(X_2)$ (hint: take $X_i$ so that they do not intersect).

**6.** (a) Find all (complex) roots of polynomial $x^4 + 1$.
   (b) Find all real values of $p$ and $q$ for which the polynomial $x^4 + 1$ is divisible by $x^2 + px + q$.