MATH 9: ASSIGNMENT 15

JAN 30, 2022

n-TH ROOT

Recall that when we multiply two complex numbers, magnitudes multiply and arguments add:

$$z_1 = r_1(\cos(\varphi_1) + i\sin(\varphi_1))$$
$$z_2 = r_1(\cos(\varphi_1) + i\sin(\varphi_2))$$
$$z_1 z_2 = r_1 r_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$$

In particular, we had de Moivre's formula: if $z = r(\cos\varphi + i\sin\varphi)$, then

$$z^n = r^n(\cos(n\varphi) + i\sin(n\varphi))$$

This also allows us to compute n-th order roots. Suppose we want to solve equation

$$z^n = w$$

where w is a given complex number with magnitude r and argument φ. Then we must have $|z|^n = |w| = r$, $n\arg z = \arg w = \varphi$, so one obvious solution is $z = \sqrt[n]{r}(\cos(\varphi/n) + i\sin(\varphi/n))$.

However, there are more solutions. Remember that arg only makes sense as a number modulo 360° (or 2π radians), so taking argument of z to be $\varphi/n + 360^\circ/n$ also works; more generally, we have solutions

$$z = \sqrt[n]{r}\left(\cos\left(\frac{\varphi + k360^\circ}{n}\right) + i\sin\left(\frac{\varphi + k360^\circ}{n}\right)\right), \quad k = 0, 1, \ldots n-1 \tag{1}$$

Altogether, this gives n solutions. This is a special case of the following extremely important result, called the **Fundamental Theorem of Algebra**.

Theorem. *Any polynomial with complex coefficients of degree n has exactly n roots (counting with multiplicities).*

There is no simple proof of this theorem (and, in fact, no purely algebraic proof: all the known proofs use some geometric arguments).

In particular, since any polynomial with real coefficients can be considered as a special case of a polynomial with complex coefficients, this shows that any real polynomial of degree n has exactly n **complex** roots.

Homework

1. Compute the following:
(a) $\dfrac{2+17i}{4-i}$
(b) $(-1+i\sqrt{3})^{2014}$
(c) $(1-i)^{-5}$

2. Solve the following equations in complex numbers. You can leave the answers in the form using sin and cos
(a) $z^2 = 1+i$
(b) $z^2+2z+2=0$
(c) $z^3 = 1$
(d) $z^2 = 1+i\sqrt{3}$
(e) $z^4 = -2$

3. On the complex plane, plot all fifth order roots of 1 and all fifth order roots of -1.

4. What is the sum of all (complex) n-th roots of 1?

5. (a) Find all roots of the polynomial $1+z+z^2+\cdots+z^n$ (Hint: remember geometric progression?)
(b) Without doing the long division, show that $1+x+x^2+\cdots+x^9$ is divisible by $1+x+\cdots+x^4$. (Hint: find the roots of each of them.)

6. Find two complex numbers satisfying $z_1+z_2=2$, $z_1z_2=5$. [Hint: they are roots of a certain quadratic polynomial.]

***7.** (a) Let $f(z)$ be a polynomial with real coefficients, and $a \in \mathbb{C}$ — a complex root of f. Assume that $a \notin \mathbb{R}$. Show that $g(z)=(z-a)(z-\overline{a})$ is a polynomial with real coefficients, and that $f(z)$ is divisible by $g(z)$.
(b) Using fundamental theorem of algebra, show that any polynomial with real coefficients can be written as a product of polynomials with real coefficients of degree at most 2.
(c) Write the polynomial x^6-1 as a product of polynomials with real coefficients of degree at most 2.