MATH 9: ASSIGNMENT 14

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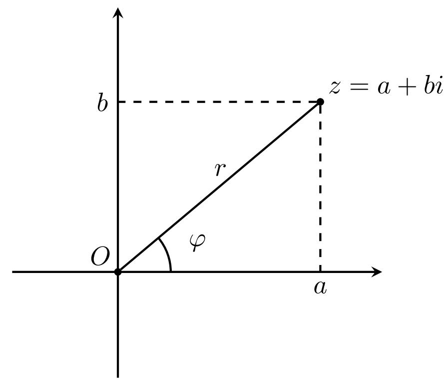
Trigonometric form of complex number



As discussed, any complex number can be described either by specifying its real and complex parts a and b, or by specifying its magnitude r and argument φ (the angle with the real axis, see picture). The relation is given by

$$z = a + bi = r(\cos\varphi + i\sin\varphi), \qquad r = |z|,\ \varphi = \arg z$$
$$a = \mathrm{Re}(z) = r\cos\varphi, \qquad b = \mathrm{Im}(z) = r\sin\varphi$$
$$r = |z| = \sqrt{a^2 + b^2}$$

Writing a complex number as $z = r(\cos\varphi + i\sin\varphi)$ is called the **trigonometric**, or **polar**, form of a complex number.

Geometric meaning of multiplication

Theorem. *If z is a complex number with magnitude r and argument φ, then multiplication by z is rotation by angle φ and dilation (rescaling) by factor r:*

$$z \cdot w = rR_\varphi(w)$$

Addition of argument

Theorem. *When we multiply two complex numbers, magnitudes multiply and arguments add:*

$$|z_1 z_2| = |z_1| \cdot |z_2|, \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \mod 360^\circ$$

Similarly,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \qquad \arg(\tfrac{z_1}{z_2}) = \arg(z_1) - \arg(z_2) \mod 360^\circ$$

Homework

1. Which transformations of the complex plane are given by the formulas

(a) $z \to iz$ (b) $z \to (1 + i\sqrt{3})z$ (c) $z \to \dfrac{z}{1+i}$

(d) $z \to \dfrac{z + \overline{z}}{2}$ (e) $z \to (1 - 2i + z)$ (f) $z \to \dfrac{z}{|z|}$

(g) $z \to i\bar{z}$ (h) $z \to -\bar{z}$

Draw the image of the square $0 \le \mathrm{Re}\, z \le 1$, $0 \le \mathrm{Im}\, z \le 1$ under each of these transformations.

2. Consider the equation $x^3 - 4x^2 + 6x - 4 = 0$.
(a) Solve this equation (hint: one of the roots is an integer).
(b) Find the sum and product of the roots in two ways: by using Vieta formulas and by explicit computation. Check that the results match.

3. Using the argument addition rule, derive a formula for $\cos(\varphi_1 + \varphi_2)$, $\sin(\varphi_1 + \varphi_2)$ in terms of sin and cos of φ_1, φ_2. [Hint: let $z_1 = \cos\varphi_1 + i\sin\varphi_1$, $z_2 = \cos\varphi_2 + i\sin\varphi_2$; then $z_1 z_2 = ?$]

4. (a) Let z be a complex number with magnitude 2 and argument 30°: $z = 2(\cos(30^\circ) + i\sin(30^\circ))$. Find the magnitude and argument of z^2; of z^3; of z^{2022}.
(b) Prove de Moivre's formula: if $z = r(\cos\varphi + i\sin\varphi)$, then

$$z^n = r^n(\cos(n\varphi) + i\sin(n\varphi))$$

5. Compute

$$(3+4i)^{-1}, \qquad (1-i)^{12}, \qquad (1-i)^{-12}, \qquad \left(\frac{1+i}{1-i}\right)^{2006}, \qquad (i\sqrt{3}-1)^{17}$$

6. Using de Moivre's formula, write a formula for $\cos(3\varphi), \sin(3\varphi)$ in terms of $\sin\varphi$, $\cos\varphi$.

7. Show that for any $n \geq 1$, $\cos(n\varphi)$ is a polynomial of $\cos\varphi$: there exists a polynomial $T_n(x)$ such that $\cos(n\varphi) = T_n(\cos(\varphi))$.

(These polynomials are called Chebyshev (or Tchebysheff) polynomials.)

***8.** Compute $1 + \cos\varphi + \cos 2\varphi + \cdots + \cos n\varphi$. [Hint: if $z = \cos\varphi + i\sin\varphi$, what is $1 + z + z^2 + \cdots + z^n$?]