MATH 9: ASSIGNMENT 13

JANUARY 16, 2022

Geometry of complex numbers

Any complex number can be written in the form $z = a + bi$, with real a, b. The number a is called *real part* of z and denoted $a = \operatorname{Re} z$; the number b is called *imaginary part* of z and denoted $b = \operatorname{Im} z$.

We can represent a complex number $z = a + bi$ by a point on the plane, with coordinates (a, b). Thus, we can identify

$$\text{complex numbers} = \text{pairs } (a, b) \text{ of real numbers} = \text{vectors in a plane}$$

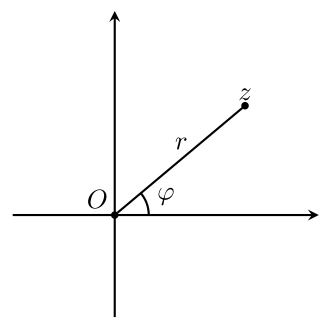
In this language, many of the operations with complex numbers have a natural geometric meaning:

- Addition of complex numbers corresponds to addition of vectors.
- The magnitude (also called absolute value) $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$ is just the distance from the corresponding point to the origin, or the length of the corresponding vector. More generally, distance between two points z, w is $|z - w|$.
- Complex conjugation $z \mapsto \bar{z}$ is just the reflection around x-axis.

The trickiest one is the multiplication. One particular case is easy: for a non-negative real number r, operation of multiplication by r is just the usual operation of multiplication of a vector by a real number: vector rz has the same direction as z but its length is multiplied by r. This operation is usually called *dilation.*

Magnitude and argument

The magnitude of a complex numbers $z = a + bi$ is $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$; geometrically it is the length of vector $z = (a, b)$. If $z \neq 0$, its *argument* $\arg z$ is defined to be the angle between the positive part of x-axis and the vector z measured counterclockwise. Thus, instead of describing a complex number by its coordinates $a = \operatorname{Re}(z)$, $b = \operatorname{Im}(z)$ we can describe it by its magnitude $r = |z|$ and argument $\varphi = \arg(z)$:



Relation between r, φ and $a = \operatorname{Re}(z)$, $b = \operatorname{Im}(z)$ is given by

$$a = r\cos(\varphi), \qquad b = r\sin(\varphi)$$
$$z = a + bi = r(\cos(\varphi) + i\sin(\varphi))$$

Geometric meaning of multiplication

Theorem.

1. *If z is a complex number with magnitude 1 and argument φ, then multiplication by z is rotation by angle φ:*
$$z \cdot w = R_\varphi(w)$$
where R_φ is operation of **counterclockwise** *rotation by angle φ around the origin.*
2. *If z is a complex number with absolute value r and argument φ, then multiplication by z is rotation by angle φ and rescaling by factor r:*
$$z \cdot w = rR_\varphi(w)$$

HOMEWORK

Throughout this assignment, we make no distinction between a point with coordinates (x, y) and a vector connecting origin $(0, 0)$ to this point.

1. Show that the operation $z \mapsto \bar{z}$ is reflection around the x axis.

2. Find the absolute value and argument of the following numbers:

$1 + i$

$-i$

$w = \frac{\sqrt{3}}{2} + \frac{i}{2}$ (hint: show that the points 0, w, $\overline{w}$ form a regular triangle)

3. Find a complex number which has argument $\pi/4 = 45^\circ$ and absolute value 2.

4. Draw the following sets of points in $\mathbb{C}$:

(a) $\{z \mid \operatorname{Re} z = 1\}$
(b) $\{z \mid |z| = 1\}$
(c) $\{z \mid \arg z = 3\pi/4\}$ (if you are not familiar with measuring angles in radians, replace $3\pi/4$ by $135°$).
(d) $\{z \mid \operatorname{Re}(z^2) = 0\}$
(e) $\{w \mid |w - 1| = 1\}$
(f) $\{w \mid |w^2| = 2\}$
(g) $\{z \mid z + \overline{z} = 0\}$

5. Show that

(a) $|\overline{z}| = |z|$, $\arg(\overline{z}) = -\arg(z)$
(b) Show that $\frac{\overline{z}}{z}$ has magnitude one. What is its argument if argument of z is φ?
(c) Check part (b) for $z = 1 + i$ by explicit calculation.

6. Let $p(x)$ be a polynomial with real coefficients.

(a) Show that for any **complex** z, we have $\overline{p(z)} = p(\overline{z})$.
(b) Show that if z is a complex root of p, i.e. $p(z) = 0$, then $\overline{z}$ is also a root.
(c) Show that if $p(z)$ has odd degree and completely factors over $\mathbb{C}$ (i.e. has as many roots as is its degree), then it must have at least one real root.

7. If z has magnitude 2 and argument $3\pi/2$, and w has absolute value 3 and argument $\pi/3$, what will be the absolute value and argument of zw? Can you write it in the form $a + bi$?

8. Let z be a complex number with magnitude 1 and argument $\pi/3$. Can you find z^3? z^6? z^{2021}?

Try doing it using as few calculations as possible.