MATH 9
ASSIGNMENT 12: COMPLEX NUMBERS
JAN 9, 2022

Complex numbers

As before, let us consider the set $\mathbb{R}[i]$ of polynomials with real coefficients in one variable (which we will now denote by i rather than x) but with one extra relation:

$$i^2 + 1 = 0.$$

Thus, we will treat two polynomials in i which differ by a multiple of $i^2 + 1$ as equal (This can be done more formally in the same was as we define multiplication and division of remainders modulo n for integers).

Note that this relation implies

$$i^2 = -1, \qquad i^3 = i^2 i = -i, \qquad i^4 = 1, \dots$$

so using this relation, any polynomial can be replaced by a polynomial of the form $a + bi$. For example,

$$(1 + i)(2 + 3i) = 2 + 4i + 3i^2 = 2 + 4i - 3 = -1 + 4i.$$

Thus, we get the following definition:

Definition. The set $\mathbb{C}$ of complex numbers is the set of expressions of the form $a+bi$, $a, b \in \mathbb{R}$, with addition and multiplication same as for usual polynomials with added relation $i^2 = -1$.

Since multiplication and addition of polynomials satisfies the usual distributivity and commutativity properties, the same holds for complex numbers.

Note that any real number a can also be considered as a complex number by writing it as $a + 0i$; thus, $\mathbb{R} \subset \mathbb{C}$.

It turns out that complex numbers can not only be multiplied and added but also divided (see problem 4).

HOMEWORK

1. Compute the following expressions involving complex numbers:

(a) $(1+2i)(3+i)$ (b) i^7

(c) $(1+i)^2$ (d) $(1+i)^7$

(e) $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3$

2. Define for a complex number $z = a + bi$ its *conjugate* by $\bar{z} = a - bi$.

(a) Prove by explicit computation that $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z} \cdot \bar{w}$.

(b) Prove that for $z = a + bi$, $z \cdot \bar{z} = a^2 + b^2$ and thus, it is non-negative real number.

3. Define for any complex number its absolute value by $|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2}$ (see previous problem). Prove that then $|zw| = |z||w|$. [Hint: use formula $|z| = \sqrt{z\bar{z}}$ instead of $|z| = \sqrt{a^2+b^2}$.]

4. Prove that any non-zero complex number z has an inverse: there exists w such that $zw = 1$ (hint: $z\bar{z} = |z|^2$).

5. Compute

(a) $(1+i)^{-1}$ (b) $\dfrac{1+i}{1-i}$

(c) $(3+4i)^{-1}$ (d) $(1+i)^{-3}$

6. (a) Find a complex number z such that $z^2 = i$

(b) Find a complex number z such that $z^2 = -2 + 2i\sqrt{3}$.

[Hint: write z in the form $z = a + bi$ and then write and solve equation for a, b]

7. Find two numbers u, v such that

$$u + v = 6$$
$$uv = 13$$

8. Find three numbers a, b, c such that

$$a + b + c = 2$$
$$ab + ac + bc = -7$$
$$abc = -14$$