MATH 9
ASSIGNMENT 8: VIETA FORMULAS

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Roots and Bezout theorem

Recall from last time:

Theorem (Bezout theorem)**.** *When a polynomial $p(x)$ is divided by $(x-c)$, the remainder is $p(c)$. In particular, $p(x)$ is divisible by $(x-c)$ if and only if c is a root.*

More generally, we have

Theorem. *If $x_1, x_2, \dots, x_k$ — distinct roots of polynomial $f(x)$, then $f(x)$ is divisible by $(x-x_1)(x-x_2)\dots(x-x_k)$.*

In particular, if $x_1, x_2, \dots, x_n$ are n distinct roots of the polynomial of degree n, then $f(x) = c(x-x_1)(x-x_2)\dots(x-x_n)$ for some constant c.

This also implies the following result.

Theorem. *A non-zero polynomial of degree n can not have more than n roots.*

Multiple roots

Definition. A number a is called a multiple root of a polynomial $f(x)$, with multiplicity m, if $f(x)$ is divisible by $(x-a)^m$ and not divisible by $(x-a)^{m+1}$.

Roots of multiplicity one are also called simple roots; of multiplicity two, double roots.

For example, polynomial $(x-1)^2(x-5)$ has a simple root $x=5$ and double root $x=1$.

The following theorem generalizes results of the previous homework:

Theorem. *If $a_1, \dots, a_k$ are distinct roots of polynomial $f(x)$, and $m_1, \dots, m_k$ are multiplicities of these roots, then $f(x)$ is divisible by the product $(x-a_1)^{m_1}\dots(x-a_k)^{m_k}$.*

It is frequently convenient, when listing roots of a polynomial, to list double root twice, triple root three times, etc, for example, listing the roots of polynomial $(x-1)^2(x-5)$ as $x_1=1$, $x_2=1$, $x_3=5$. This is called "listing the roots with multiplicities". Then the previous result can be rewritten as follows:

Theorem. *If $x_1, \dots, x_n$ are roots of polynomial $f(x)$, listed with multiplicities, then $f(x)$ is divisible by $(x-x_1)\dots(x-x_n)$.*

Integer roots

Explicitly finding roots of a polynomial is very hard. However, the following result can sometimes help.

Theorem. *If $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ is a polynomial with integer coefficients and leading coefficient 1, and $c = p/q$ is a rational root of f, then c must be integer; moreover, c must be a divisor of the constant term a_0.*

Proof of this theorem is given in problem 3 in the homework.

Note however that a polynomial with integer coefficients can have irrational roots, and this theorem doesn't give any information about them.

Vieta formulas

Let $f(x)$ be a polynomial of degree n with leading coefficient 1 and roots $x_1, x_2, \dots, x_n$; then

$$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_x + a_0 = (x - x_1)(x - x_2)\dots(x - x_n)$$

This shows that the coefficients of $f(x)$ can be written in terms of roots (just opening the parentheses in the right hand side and collecting the like terms). The resulting formulas are called *Vieta formulas*. Here they are for $n = 2$:

$$(x - x_1)(x - x_2) = x^2 + px + q, \qquad q = x_1x_2, \qquad p = -(x_1 + x_2)$$

and for $n = 3$:

$$\begin{aligned}
&(x - x_1)(x - x_2)(x - x_3) = x^3 + a_2x^2 + a_1x + a_0,\\
&a_0 = -x_1x_2x_3\\
&a_1 = x_1x_2 + x_1x_3 + x_2x_3\\
&a_2 = -(x_1 + x_2 + x_3)
\end{aligned}$$

Problems

1. (a) Show that $x^{99} - 1$ is divisible by $x - 1$, by $x^3 - 1$, by $x^{11} - 1$.
(b) Can you find any factors of the number $179^{57} - 1$?

2. Let $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ be a polynomial with integer coefficients.
(a) Show that if $c = \frac{p}{q}$ is a rational root of this polynomial, then $q = 1$, i.e. a must be integer. [Hint: which of the terms in $q^{n-1}f(c)$ are integer?]
(b) Show that one can write $f(x) = (x-c)g(x)$, where $g(x)$ is a polynomial with integer coefficients.
(c) Show that c is a divisor of a_0.

3. Find roots with multiplicities of the following polynomials. Factor these polynomials if possible.
$x^3 - 3x^2 + 4$
$x^4 - 5x^3 + 6x^2$
$x^3 + 4x^2 - x - 10$

4. Let x_1, x_2 be roots of polynomial $x^2 + px + q$. Without using the explicit formula for x_1, x_2, express the following quantities in terms of p, q:
(a) $(x_1 + x_2)^2$
(b) $x_1^2 + x_2^2$ [Hint: what is the difference between this and $(x_1 + x_2)^2$?]
(c) $(x_1 - x_2)^2$
(d) $x_1^3 + x_2^3$ [Hint: similar to part (b).]
(e) $\frac{1}{x_1} + \frac{1}{x_2}$

5. Let x_1, x_2, x_3 be roots of polynomial $f(x) = x^3 - 5x + 11$. Find the following quantities:
(a) $x_1 + x_2 + x_3$
(b) $x_1x_2x_3$
(c) $x_1^2 + x_2^2 + x_3^2$
(d) Write a cubic polynomial whose roots are x_1^2, x_2^2, x_3^2.
Note: you are not asked to find x_1, x_2, x_3 — this is hard.

6.
(a) Prove Vieta formulas for $n = 3$
*(b) Do you see the pattern? Can you guess Vieta formulas for $n = 4$?

7. It is known that numbers a, b, c satisfy $a + b + c > 0$, $ab + bc + ac > 0$, $abc > 0$. Prove that each of numbers a, b, c is positive. [Hint: consider polynomial $(x + a)(x + b)(x + c)$; what can you say about the signs of coefficients and roots of this polynomial?]