MATH 9
ASSIGNMENT 5: MATHEMATICAL INDUCTION

October 31, 2021

The **Principle of Mathematical Induction** is an extremely famous and useful mathematical argument, first formalized by mathematician Peano, in what was also the first formalization of modern logic. It's the idea that if you start at 0 and keep hopping upwards by +1, that eventually you will reach any natural number. Formally, it is as follows:

Let $P(n)$ be a statement which depends on a natural number n (natural numbers are nonnegative integers). Suppose that we know the following:

- $P(0)$ is true
- For every n, the statement $\Big(P(n) \implies P(n+1)\Big)$ is true.

Then $P(n)$ is true for all n.

Proving that $P(0)$ is true is called the **base case**.

Proving the implication $P(n) \implies P(n+1)$ is called the **inductive step**. It is important to understand that it is the implication itself you are proving, not either of the statements $P(n)$ or $P(n+1)$. In other words, you are proving that **if** $P(n)$ is true, **then** $P(n+1)$ is also true.

A variation of the principle of mathematical induction is when instead of taking the base case to be $n = 0$, you take the base case $n = 1$ (or some other number n_0); in this case, mathematical induction establishes that the statement is true for all $n \geq n_0$.

Homework

1. Use mathematical induction to prove the following formulas
(a) $1 + 2 + 3 + \dots n = \frac{n(n+1)}{2}$
(b) $1 + 3 + 5 + \dots (2n-1) = n^2$
(c) $1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

2. Guess a formula for the product

$$\left(1 - \tfrac{1}{2^2}\right)\left(1 - \tfrac{1}{3^2}\right)\dots\left(1 - \tfrac{1}{n^2}\right)$$

and prove it using induction. [Hint: try computing the answer for $n = 2, 3, 4, 5$ and writing it as a fraction with denominator $2n$; see if you can guess the pattern.]

3. Guess a formula for the product

$$\left(1 - \tfrac{1}{2^2}\right)\left(1 - \tfrac{1}{3^2}\right)\dots\left(1 - \tfrac{1}{n^2}\right)$$

and prove it using induction. [Hint: try computing the answer for $n = 2, 3, 4, 5$ and writing it as a fraction with denominator $2n$; see if you can guess the pattern.]

4. Use mathematical induction to prove that for any real $x > -1$ and integer $n \geq 1$, we have

$$(1 + x)^n > 1 + nx$$

5. Let the numbers $\binom{n}{k}$ be defined for all $n \geq 0$ and arbitrary integer k by the following rules:
For $n = 0$:

$$\binom{0}{0} = 1, \quad \binom{0}{k} = 0 \qquad \text{for all } k \neq 0$$

For positive n:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

(Of course, you recognize that these are the rules of Pascal triangle.)

Prove that then, for all $k \in \mathbb{Z}$ and $n \geq 0$ we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

6. Can you explain what is wrong with this argument?

Claim: in any group of $n \geq 1$ horses, all of them have the same color.

Proof: by induction in n

Base case: $n = 1$ is obvious.

Induction step: If we have a group of $n+1$ horses, choose two subgroups: one consisting of horses 1,2, ..., n; the other, horses 2, 3, ..., $n+1$. By induction assumption, in each subgroup horses have the same color. On the other hand, these two subgroups have horses in common; thus, all $n+1$ horses have the same color.

Conclusion: all horses existing in the world have the same color.