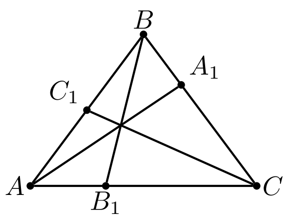
MATH 9
ASSIGNMENT 3: CEVA THEOREM AND MORE
OCT 17, 2021

Ceva theorem

Theorem. *Let points A_1, B_1, C_1 be on the sides BC, AC, AB respectively of triangle ABC. Then lines AA_1, BB_1, CC_1 intersect at a single point if and only if*

$$\frac{CA_1}{BA_1} \cdot \frac{BC_1}{AC_1} \cdot \frac{AB_1}{CB_1} = 1$$

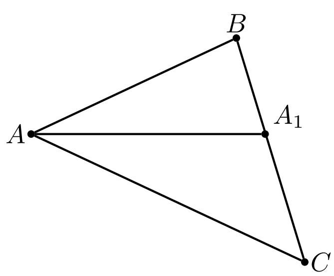


Problems

1. Let $ABCD$ be a convex quadrilateral, and let points K, L, M, N be midpoints of sides AB, BC, CD, and DA respectively. Let O be the intersection point of segments KM and LN. Prove that O is the midpoint of each of them.

2. Let AA_1 be the bisector of angle A in triangle ABC. Show that then, A_1 divides side BC in the proportion equal to ratio of sides $c = AB$, $b = AC$:

$$|BA_1| : |CA_1| = c : b$$



[Hint: drop perpendiculars BB_1, CC_1 from vertices B, C to line AA_1. Use similar triangles to show that $BB_1 : CC_1 = c : b$.]

3. Use previous problem and Ceva theorem to give another proof of the fact that the three angle bisectors of a triangle intersect at a single point.

***4.** Show that the intersection point of the three angle bisectors in a triangle is the center of mass of the three sides (not vertices!) of the triangle, if we think of each side as a thin rod, with uniform density, so that the mass of a rod is proportional to its length.

5. Let A, B be distinct points. Show that then a point M lies on line AB if and only if one can write

$$\overrightarrow{OM} = t\,\overrightarrow{OA} + (1-t)\,\overrightarrow{OB}$$

fro some real t.

In particular, M lies on **segment** AB if and only if it can be written in the form above with $0 \le t \le 1$.

[Hint: write $\overrightarrow{OM}$ as combination of $\overrightarrow{OA}$ and $\overrightarrow{AB}$.]

6. Points A, B are moving along sides of an angle XOY so that the quantity

$$\frac{p}{OA} + \frac{q}{OB}$$

stays constant. [Here p, q are some fixed positive real numbers.]

Prove that then there is a point M inside this angle such that for all such positions of A, B, the line AB goes through M.

7. Let R be the operation of rotation by angle $60°$: for any vector $\vec{v}$, $R\vec{v}$ is a vector of same length as $\vec{v}$ but rotated 60 degrees counterclockwise.

(a) Show that $R(c\vec{v}) = cR(\vec{v})$, $R(\vec{v} + \vec{w}) = R(\vec{v}) + R(\vec{w})$.

(b) Given a triangle ABC, build on the outside of side AB an equilateral triangle. Denote by C_1 the center of this triangle.

Write vector $\overrightarrow{OC_1}$ as combination of $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and operation R.

*(c) Repeat the same operation with two other sides of triangle ABC, building equilateral triangles on sides BC, AC. Show that the centers of the three constructed triangles themselves form an equilateral triangle.

[Hint: it suffices to show that $R(\overrightarrow{A_1B_1}) = \overrightarrow{B_1C_1}$.]