MATH 9
ASSIGNMENT 2: CENTER OF MASS

OCT 3, 2021

Center of mass

Let $A_1, \dots, A_n$ be a collection of points and $m_1, \dots, m_n$ — some real numbers, representing masses placed at these points. Then the center of mass of such a collection of masses is defined to be a point M such that

$$\overrightarrow{OM} = \frac{m_1 \overrightarrow{OA_1} + \dots + m_n \overrightarrow{OA_n}}{m_1 + \dots + m_n}$$

where O is the origin.

Problems

1. (a) Let M be a point on the segment A_1A_2 which divides this segment in proportion $MA_1 : MA_2 = 5 : 7$. Show that then, M is the same as the center of mass of the system consisting of mass $m_1 = 7$ at point A_1 and mass $m_2 = 5$ at point A_2. [Hint: compare with problem 2 from previous homework.]

(b) Show that the center of mass of system of two points A_1, A_2 with masses m_1, m_2 is the point on the segment A_1A_2, which divides this segment in proportion $MA_1 : MA_2 = m_2 : m_1$. In particular, if $m_1 = m_2$, then this point is the midpoint of A_1A_2.

2. Show that if M is the center of mass of points $A_1, \dots, A_n$, then for any point X (not only for the origin), we have

$$\overrightarrow{XM} = \frac{m_1 \overrightarrow{XA_1} + \dots + m_n \overrightarrow{XA_n}}{m_1 + \dots + m_n}$$

(hint: $\overrightarrow{XM} = \overrightarrow{XO} + \overrightarrow{OM}$).

3. Show that the center of mass of some collection of points doesn't change if we replace two points A_1, A_2 with masses m_1, m_2 by a single mass $m_1 + m_2$ placed at the center of mass of A_1, A_2.

4. Let M be the center of mass of a system of 3 points A, B, C with equal masses. Show that then M lies on the median AA_1, dividing it in proportion $2 : 1$. Deduce from this that in fact, all three medians of a triangle pass through M (and thus intersect at a single point).

5. On each side of a parallelogram $ABCD$, mark a point which divides it in the proportion 2:1 (going clockwise). Prove that the marked points themselves form a parallelogram.

[Hint: denote $\overrightarrow{AB} = \vec{v}$, $\overrightarrow{AD} = \vec{w}$, and write vectors $\overrightarrow{AA_1}, \overrightarrow{AB_1}, \overrightarrow{A_1B_1}, \dots$ as combinations of $\vec{v}$, $\vec{w}$]

6. (a) In a triangle ABC, let point M_1 be on the side BC dividing it so that $M_1B : M_1C = 2 : 3$, and M_2, M_3 on sides AC, AB respectively so that

$$M_2A : M_2C = 2 : 5$$
$$M_3A : M_3B = 3 : 5$$

Prove that lines AM_1, BM_2, CM_3 intersect at a single point. [Hint: place appropriate masses at points A, B, C]

(b) Prove Ceva theorem: if points M_1, M_2, M_3 are on the sides BC, AC, AB respectively of triangle ABC, then lines AM_1, BM_2, CM_3 intersect at a single point if and only if

$$\frac{CM_1}{BM_1} \cdot \frac{BM_3}{AM_3} \cdot \frac{AM_2}{CM_2} = 1$$

[Hint: place appropriate masses at points A, B, C]