Classwork: Symmetries and Group Theory

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Definition. A group G is a set with a binary operation $*$ such that,
(1) Closure: $\forall a, b \in G \Rightarrow a * b \in G$.
(2) Identity: $\exists e \in G$ such that $\forall a \in G, e * a = a * e = a$.
(3) Inverse: $\forall a \in G, \exists a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.
(4) Associativity: $\forall a, b, c \in G \Rightarrow (a * b) * c = a * (b * c)$.

Definition. A group $(G, *)$ is called Abelian if $\forall a, b \in G \Rightarrow a * b = b * a$.

Examples. (1) G is the set $\{0, 1\}$ with the binary operation $*$ being addition modulo 2.
(2) $(G, *) = (Z, +)$, the set of integers with addition.
(3) The set $S = \{0, 1, ..., n-1\}$ *is not* a group with respect to multiplication modulo n since 0 has no inverse.
(4) The rotation and reflection symmetries of a regular n-polygon form a group, the Dihedral group D_n. It is non-abelian.

Definition. The order of the group is the number of its elements.

Examples. (1) The order of the group D_n is $2n$, n rotations by angles $\frac{2\pi}{k}, k = 0, ..., n-1$ and n reflections.
(2) The order of the permutation group of n elements S_n is $n!$.

Definition. A cyclic group C_n is generated by powers of one element $a \in C_n$ $C_n = \{a^0, a^1, ..., a^{n-1}\}$.

Example. Rotational symmetries of the square $C_4 = \{R_0, R_{\frac{\pi}{2}}, R_\pi, R_{\frac{3\pi}{2}}\}$.

Definition. A subgroup H of a group $(G, *)$ is a subset $H \subset G$ that forms a group with respect to the same binary operation $*$.

Example. Rotations form a subgroup of the Dihedral group. Reflections do not form a subgroup since the composition of two different reflections results in a rotation.

Definition. Given two groups $(G, *)$ and $(H, \bullet)$, a group isomorphism is a bijection $f : G \to H$ such that $f(a * b) = f(a) \bullet f(b)$.

Examples. (1) The symmetry group D_3 of the equilateral triangle is isomorphic to the permutation group of three elements S_3
(2) The symmetry group D_4 of the square *is not* isomorphic to the permutation group of four elements S_4.