Homework for January 30, 2022

Algebra.

Review the last algebra classwork handouts. Solve the unsolved problems from the previous homeworks. Try solving the following problems.

1. Assume that the set of rational numbers $\mathbb{Q}$ is divided into two subsets, $\mathbb{Q}_<$ and $\mathbb{Q}_>$, such that all elements of $\mathbb{Q}_>$ are larger than any element of $\mathbb{Q}_<$: $\forall a \in \mathbb{Q}_<, \forall b \in \mathbb{Q}_>, a < b$.
    a. Prove that if $\mathbb{Q}_>$ contains the smallest element, $\exists b_0 \in \mathbb{Q}_>, \forall b \in \mathbb{Q}_>, b_0 \leq b$, then $\mathbb{Q}_<$ does not contain the largest element
    b. Prove that if $\mathbb{Q}_<$ contains the largest element, $\exists a_0 \in \mathbb{Q}_<, \forall a \in \mathbb{Q}_<, a \leq a_0$, then $\mathbb{Q}_>$ does not contain the smallest element
    c. Present an example of such a partition, where neither $\mathbb{Q}_>$ contains the smallest element, nor $\mathbb{Q}_<$ contains the largest element
2. Prove the following properties of countable sets. For any two countable sets, A, B,
    a. Union, $A \cup B$, is also countable, $\left((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\right) \Rightarrow (c(A \cup B) = \aleph_0)$
    b. Product, $A \times B = \{(a, b), a \in A, b \in B\}$, is also countable, $\left((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\right) \Rightarrow (c(A \cup B) = \aleph_0)$
    c. For a collection of countable sets, $\{A_n\}, c(A_n) = \aleph_0$, the union is also countable, $c(A_1 \cup A_2 \ldots \cup A_n) = \aleph_0$

3. Let W be the set of all "words" that can be written using the alphabet consisitng of 26 lowercase English letters; by a "word", we mean any (finite) sequence of letters, even if it makes no sense – for example, abababaaaaa. Prove that W is countable. [Hint: for any n, there are only finitely many words of length n.]
4. Compare the following real numbers (are they equal? which is larger?)
    a. 1.33333… = 1.(3) and 4/3
    b. 0.09999… = 0.0(9) and 1/10
    c. 99.9999… = 99.(9) and 100
    d. $\sqrt[2]{2}$ and $\sqrt[3]{3}$
5. Simplify the following real numbers. Are these numbers rational? (hint: you may use the formula for an infinite geometric series).

    a. 1/1.1111...=1/1.1(1)
    b. 2/1.2323...=2/1.23(23)
    c. 3/0.123123...=3/0.123(123)
6. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).
    a. 1/8
    b. 2/7
    c. 0.1
    d. 0.33333... = 0.(3)
    e. 0.13333... = 0.1(3)
7. Try proving the following properties of real numbers and arithmetical operations on them using definition of a real number as the Dedekind section and the validity of these properties for rational numbers.

Ordering and comparison.

1. $\forall\ a, b \in \mathbb{R}$, one and only one of the following relations holds
    - $a = b$
    - $a < b$
    - $a > b$
2. $\forall\ a, b \in \mathbb{R}, \exists c \in \mathbb{R}, (c > a) \wedge (c < b)$, i.e. $a < c < b$
3. Transitivity. $\forall\ a, b, c \in \mathbb{R}, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
4. Archimedean property. $\forall\ a, b \in \mathbb{R}, a > b > 0, \exists n \in \mathbb{N}$, such that $a < nb$

Addition and subtraction.

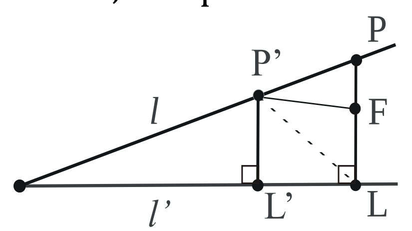
- $\forall\ a, b \in \mathbb{R}, a + b = b + a$
- $\forall\ a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$
- $\forall\ a \in \mathbb{R}, \exists 0 \in \mathbb{R}, a + 0 = a$
- $\forall\ a \in \mathbb{R}, \exists -a \in \mathbb{R}, a + (-a) = 0$
- $\forall\ a, b \in \mathbb{R}, a - b = a + (-b)$
- $\forall\ a, b, c \in \mathbb{R}, (a < b) \Rightarrow (a + c < b + c)$

Geometry.

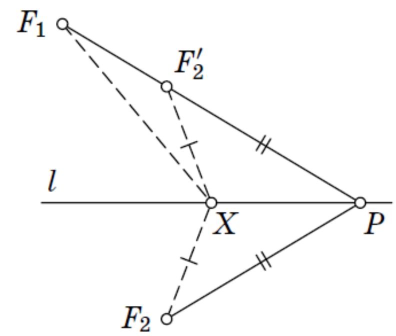
Review the previous classwork notes. Solve the problems below and the remaining problems from the previous homework.

Problems.

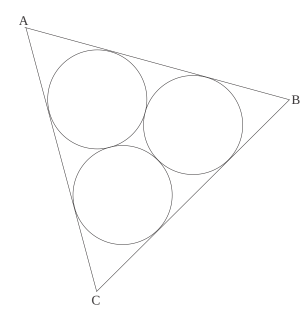
1. Given two lines, l and l', and a point F not on any of those lines, find point P on l such that the (signed) difference of distances from it to l' and F, $|P'L'| - |P'F|$, is maximal. As seen in the figure, for any P' on l the distance to l', $|P'L'| \leq |P'L| \leq |P'F| + |FL|$, where $|FL|$ is the distance from F to l'. Hence, $|P'L'| - |P'F| \leq |FL|$, and the difference is largest ($= |FL|$) when point P belongs to the perpendicular FL from point F to l'.



2. Given line l and points F_1 and F_2 lying on different sides of it, find point P on the line l such that the absolute value of the difference in distances from P to points F_1 and F_2 is maximal. As above, let $F_2{}'$ be the reflection of F_2 in l. Then for any point X on l, $|XF_2| - |XF_1'| \leq |F_1 F_2'|$.



3. Find the (x, y) coordinates of the common (intersection) point of the two lines, one passing through the origin at 45 degrees to the X-axis, and the other passing through the point (1,0) at 60 degrees to it.
4. Find the (x, y) coordinates of the common (intersection) points of the parabola $y = x^2$ and of the ellipse centered at the origin and with major axis along the Y-axis whose length equals 2, and the minor axis along the X-axis whose length equals 1.
5. (Skanavi 10.122) Find the locus of the midpoints of all chords of a given circle with the center O, which intersect given chord AB of this circle.



6. Three circles of radius r touch each other. Find the area of the triangle ABC formed by tangents to pairs of circles (see figure).