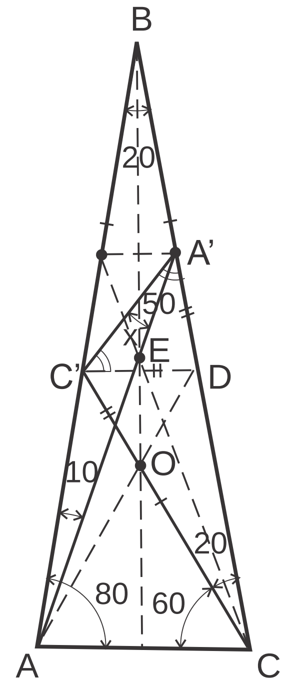
December 19, 2021

Geometry.

<u>Solutions to some homework problems</u>.

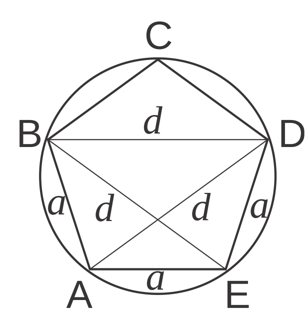


1. **Problem**. In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 20°$ and $\angle BAA' = 10°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.

   Solution. Making supplementary constructs shown in the figure, we see that $\Delta DOC'$ is equilateral (all angles are $60°$), while $|OD| = |C'O| = |A'D|$ as corresponding elements in congruent triangles, $\Delta BOC' \cong \Delta BOD \cong \Delta AA'D$. Hence, $\Delta A'C'D$ is isosceles ns $\angle A'C'D = \angle CA'D = 50°$, wherefrom, $x = \angle CA'D - \angle AA'D = 50° - 30° = 20°$.

2. **Problem**. In a triangle ABC, Cevian segments AA', BB' and CC' are concurrent and cross at a point M (point C' is on the side AB, point B' is on the side AC, and point A' is on the side BC). Given the ratios $\frac{AC'}{C'B} = p$ and $\frac{AB'}{B'C} = q$, find the ratio $\frac{AM}{MA'}$ (express it through p and q).

   Solution. Load vertices A, B and C with masses $m_A = 1$, $m_B = p$, and $m_C = q$, respectively. This makes point C' is on the side AB center of mass for $m_A = 1$ and $m_B = p$ and point B' is on the side AC center of mass for $m_A = 1$ and $m_C = q$. Point M is then the center of mass for all three masses. Moving masses $m_B = p$, and $m_C = q$ to their center of mass A' is on the side BC and using the lever rule, we obtain, $\frac{AM}{MA'} = p + q$.



3. **Problem**. Using the Ptolemy's theorem, prove the following:
   b. In a regular pentagon, the ratio of the length of a diagonal to the length of a side is the golden ratio, ϕ.

   Solution. Consider the circumscribed circle (why can be circle circumscribed around a regular pentagon?). A diagonal divides pentagon into an isosceles triangle and a quadrilateral (trapezoid) inscribed in the same circle. The sides of the trapezoid are a, a, a, and d, where a is the side of the pentagon and d its diagonal. Applying the Ptolemy's theorem we obtain, $a \cdot a + a \cdot d = d \cdot d$, or, $\left(\frac{d}{a}\right)^2 - \left(\frac{d}{a}\right) - 1 = 0$, which is the equation for the golden ratio, $\frac{d}{a} = \frac{1+\sqrt{5}}{2}$.