October 31, 2021

Algebra.

Principle of Mathematical Induction (continued).

Arithmetic and geometric mean inequality: Proof by induction.

The **arithmetic mean** of n numbers, $\{a_1, a_2, \dots, a_n\}$, is, by definition,

$$A_n = \frac{a_1 + a_2 + \dots + a_n}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i \qquad (1)$$

The **geometric mean** of n non-negative numbers, $\{a_n \geq 0\}$, is, by definition,

$$G_n = \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} = \sqrt[n]{\prod_{i=1}^{n} a_i} \qquad (2)$$

Theorem. For any set of n non-negative numbers, the arithmetic mean is not smaller than the geometric mean,

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} \qquad (3)$$

The standard proof of this fact by mathematical induction is given below.

Induction basis. For $n = 1$ the statement is a true equality. We can also easily prove that it holds for $n = 2$. Indeed, $(a_1 + a_2)^2 - 4a_1a_2 = (a_1 - a_2)^2 \geq 0$ $\Rightarrow a_1 + a_2 \geq 2\sqrt{a_1a_2}$.

Induction hypothesis. Suppose the inequality holds for any set of n non-negative numbers, $\{a_1, a_2, \dots, a_n\}$.

Induction step. We have to prove that the inequality then also holds for any set of $n + 1$ non-negative numbers, $\{a_1, a_2, \dots, a_{n+1}\}$.

Proof. If $a_1 = a_2 = \cdots = a_n = a_{n+1}$, then the equality, $A_{n+1} = G_{n+1}$, obviously holds. If not all numbers are equal, then there is the smallest (smaller than the mean) and the largest (larger than the mean). Let these be $a_{n+1} < A_{n+1}$, and $a_n > A_{n+1}$. Consider new sequence of n non-negative numbers, $\{a_1, a_2, \dots, a_{n-1}, a_n + a_{n+1} - A_{n+1}\}$. The arithmetic mean for these n numbers is still equal to A_{n+1},

$$\frac{a_1+a_2+\cdots+a_{n-1}+a_n+a_{n+1}-A_{n+1}}{n} = \frac{n+1}{n}A_{n+1} - \frac{1}{n}A_{n+1} = A_{n+1} \quad (4)$$

Therefore, by induction hypothesis,

$$(A_{n+1})^n \geq a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} \cdot (a_n + a_{n+1} - A_{n+1}) \quad (5)$$

$$(A_{n+1})^{n+1} \geq a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} \cdot (a_n + a_{n+1} - A_{n+1}) \cdot A_{n+1} \quad (6)$$

Wherein, using $a_{n+1} < A_{n+1}$ and $a_n > A_{n+1}$, as assumed above, we get $(a_n - A_{n+1})(A_{n+1} - a_{n+1}) > 0$, or, $a_n a_{n+1} < (a_n + a_{n+1} - A_{n+1})A_{n+1}$, so we could substitute the last two terms in the product with $a_n \cdot a_{n+1}$, while keeping the inequality. This completes the proof. ¤

Newton's binomial.

The **Newton's binomial** is an expression representing the simplest n-th degree factorized polynomial of two variables, $P_n(x, y) = (x + y)^n$ in the form of the polynomial summation (i.e. expanding the brackets),

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + \binom{n}{n-1}x\,y^{n-1} + \binom{n}{n}y^n, \quad (1a)$$

$$(x+y)^n = C_n^0 x^n + C_n^1 x^{n-1}y + C_n^2 x^{n-2}y^2 + \cdots + C_n^k x^{n-k}y^k + \cdots + C_n^{n-1}x\,y^{n-1} + C_n^n y^n. \quad (1b)$$

For $n = 1, 2, 3, \ldots$, these are familiar expressions,

$$(x + y) = x + y,$$

$$(x + y)^2 = x^2 + 2xy + y^2,$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

etc.

The Newton's binomial formula could be established either by directly expanding the brackets, or proven using the mathematical induction.

Exercise. Prove the Newton's binomial using the mathematical induction.

Induction basis. For $n = 1$ the statement is a true equality, $(x + y)^1 = C_1^0 x + C_1^1 y$. We can also easily prove that it holds for $n = 2$. Indeed, $(x + y)^2 = C_2^0 x^2 + C_2^1 xy + C_2^2 y^2$.

Induction hypothesis. Suppose the equality holds for some $n \in N$, that is,

$$(x + y)^n = C_n^0 x^n + C_n^1 x^{n-1} y + C_n^2 x^{n-2} y^2 + \cdots + C_n^k x^{n-k} y^k + \cdots + C_n^{n-1} x\, y^{n-1} + C_n^n y^n$$

Induction step. We have to prove that it then also holds for the next integer, $n + 1$,

$$(x + y)^{n+1} = C_{n+1}^0 x^{n+1} + C_{n+1}^1 x^n y + C_{n+1}^2 x^{n-1} y^2 + \cdots + C_{n+1}^k x^{n+1-k} y^k + \cdots + C_{n+1}^n x\, y^n + C_{n+1}^{n+1} y^{n+1}$$

Proof. $(x + y)^{n+1} = (x + y)^n (x + y) =$

$$(C_n^0 x^n + C_n^1 x^{n-1} y + C_n^2 x^{n-2} y^2 + \cdots + C_n^k x^{n-k} y^k + \cdots + C_n^{n-1} x\, y^{n-1} + C_n^n y^n)(x + y) =$$

$$C_n^0 x^{n+1} + C_n^1 x^n y + C_n^2 x^{n-1} y^2 + \cdots + C_n^k x^{n-k+1} y^k + \cdots + C_n^{n-1} x^2\, y^{n-1} + C_n^n x y^n + C_n^0 x^n y + C_n^1 x^{n-1} y^2 + C_n^2 x^{n-2} y^3 + \cdots + C_n^k x^{n-k} y^{k+1} + \cdots + C_n^{n-1} x\, y^n + C_n^n y^{n+1} =$$

$$C_n^0 x^{n+1} + (C_n^1 + C_n^0) x^n y + (C_n^2 + C_n^1) x^{n-1} y^2 + \cdots + (C_n^k + C_n^{k-1}) x^{n-k+1} y^k + \cdots + (C_n^n + C_n^{n-1}) x\, y^n + C_n^n y^{n+1} =$$

$$C_{n+1}^0 x^{n+1} + C_{n+1}^1 x^n y + C_{n+1}^2 x^{n-1} y^2 + \cdots + C_{n+1}^k x^{n+1-k} y^k + \cdots + C_{n+1}^n x\, y^n + C_{n+1}^{n+1} y^{n+1},$$

Where we have used the property of binomial coefficients, $C_n^k + C_n^{k-1} = C_{n+1}^k$.
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Properties of binomial coefficients

Binomial coefficients are defined by

$$C_n^k = {}_nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial coefficients have clear and important combinatorial meaning.

- There are $\binom{n}{k}$ ways to choose k elements from a set of n elements.
- There are $\binom{n+k-1}{k}$ ways to choose k elements from a set of n if repetitions are allowed.
- There are $\binom{n+k}{k}$ strings containing k ones and n zeros.
- There are $\binom{n+1}{k}$ strings consisting of k ones and n zeros such that no two ones are adjacent.

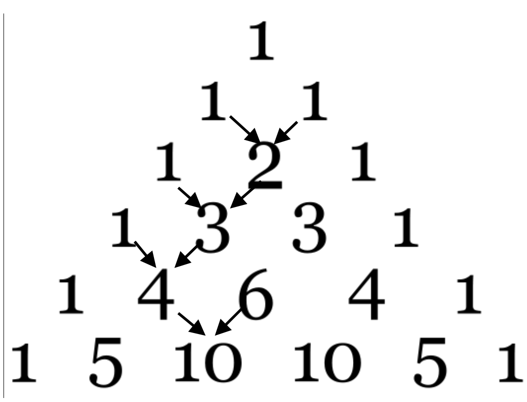
They satisfy the following identities,

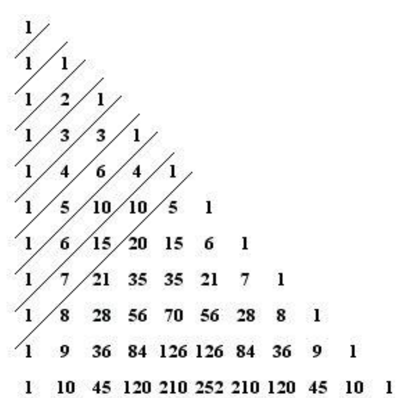
$$C_{n+1}^{k+1} = C_n^k + C_n^{k+1} \Leftrightarrow \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

$$C_{n+1}^{k} = C_n^k + C_n^{k-1} \Leftrightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\sum_{k=0}^{n} C_n^k = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

<u>Patterns in the Pascal triangle</u>





$C_n^k = C_{n-1}^{k-1} + C_{n-1}^{k}$	Fibonacci numbers (sum of the "shallow" diagonals:

Exercise. Find the sum of the top n rows in the Pascal triangle, $\sum_{m=0}^{n}(\sum_{k=0}^{m} C_m^k) = 2^{n+1} - 1$.

Review of selected homework problems.

Problem 4. Using mathematical induction, prove that

a. $P_n: \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution.

Basis: $P_1: \sum_{k=1}^{1} k^2 = 1 = \frac{1\cdot(1+1)\cdot(2\cdot 1+1)}{6}$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1}: \sum_{k=1}^{n+1} k^2 = 1^2 + 2^2 + 3^2 + \cdots + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$

Proof: $\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)}{6}(n(2n+1) + 6n + 6) = \frac{(2n+1)(2n^2+7n+6)}{3} = \frac{(n+1)(n+2)(2n+3)}{6}$,

where we used the induction hypothesis, P_n, to replace the sum of the first n terms with a formula given by P_n. ¤

b. $P_n: \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Solution.

Basis: $P_1: \sum_{k=1}^{1} k^3 = 1 = \left[\frac{1(1+1)}{2}\right]^2$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1}: \sum_{k=1}^{n+1} k^3 = 1^3 + 2^3 + 3^3 + \cdots + (n+1)^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2$

Proof: $\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 = \left[\frac{(n+1)}{2}\right]^2 (n^2 + 4n + 4) = \left[\frac{(n+1)(n+2)}{2}\right]^2$, where we used the induction hypothesis, P_n, to replace the sum of the first n terms with a formula given by P_n. ¤

c. $P_n: \sum_{k=1}^{n} \frac{1}{k^2+k} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$

Solution.

Basis: $P_1: \sum_{k=1}^{1} \frac{1}{k^2+k} = \frac{1}{2} = \frac{1}{1+1}$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1}: \sum_{k=1}^{n+1} \frac{1}{k^2+k} = \frac{n+1}{n+2}$

Proof: $\sum_{k=1}^{n+1} \frac{1}{k^2+k} = \sum_{k=0}^{n} \frac{1}{k^2+k} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{n+1}{n+2}$. ¤

e. $P_n: \forall n, \exists k, 5^n + 3 = 4k$

Solution.

Basis: $P_1: n = 1, \exists k, 5^1 + 3 = 8 = 4k \Leftrightarrow k = 2$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1}: \forall n, \exists q, 5^{n+1} + 3 = 4q$

Proof: $5^{n+1} + 3 = 5 \cdot 5^n + 3 = 5 \cdot (4k - 3) + 3 = 5 \cdot 4k - 12 = 4 \cdot (5k - 3)$.

Where we used the induction hypothesis, P_n, to replace 5^n with a formula, $5^n = 4k - 3$, given by P_n. ¤

e. $P_n: \forall n \geq 2, \forall x > -1, \ \ (1 + x)^n \geq 1 + nx$

Solution.

Basis: $P_2: \forall x > -1, n = 2, \ \ (1 + x)^2 = 1 + 2x + x^2 \geq 1 + 2x$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1}: \forall n \geq 2, \forall x > -1, \ (1 + x)^{n+1} \geq 1 + (n + 1)x$

Proof: $(1 + x)^{n+1} = (1 + x)\,(1 + x)^n \geq (1 + x)(1 + nx) = 1 + (n + 1)x + x^2 \geq 1 + (n + 1)x$. ¤