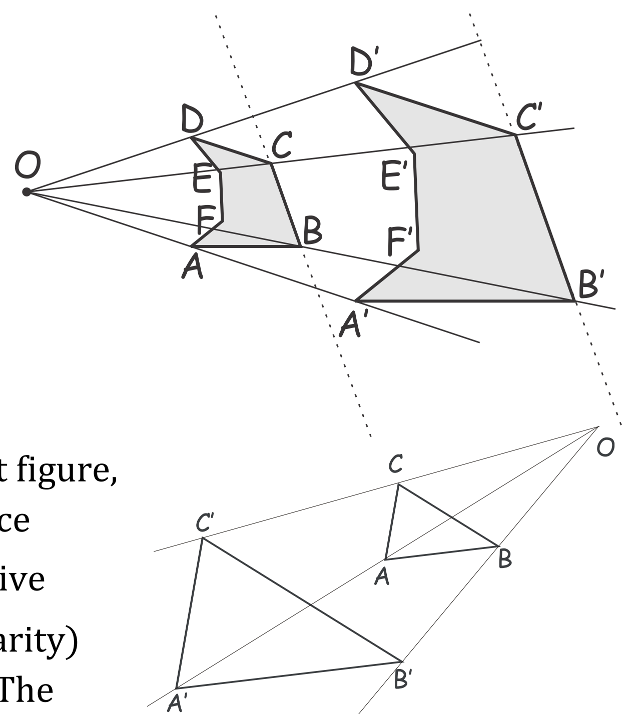
October 24, 2021

Geometry.

Similarity and homothety.

Recap: Similarity and homothety.

Homothety



Definition. Two figures are homothetic with respect to a point O, if for each point A of one figure there is a corresponding point A' belonging to the other figure, such that A' lies on the line (OA) at a distance $|OA'| = k|OA|$ ($k > 0$) from point O, and vice versa, for each point A' of the second figure there is a corresponding point A belonging to the first figure, such that A' lies on the line (OA) at a distance $|OA| = \frac{1}{k}|OA'|$ from point O. Here the positive number k is called the homothety (or similarity) coefficient. Homothetic figures are **similar**. The transformation of one figure (e.g. multilateral $ABCDEF$) into the figure $A'B'C'D'E'F'$ is called homothety, or similarity transformation.

Thales Theorem Corollary 1. The corresponding segments (e.g. sides) of the homothetic figures are parallel.

Thales Theorem Corollary 2. The ratio of the corresponding elements (e.g. sides) of the homothetic figures equals k.

Exercise. What is the ratio of the areas of two similar (homothetic) figures?

Definition. Consider triangles, or polygons, such that angles of one of them are congruent to the respective angles of the other(s). Sides which are adjacent to the congruent angles are called *homologous*. In triangles, sides opposite to the congruent angles are also homologous.

Definition. Two triangles are similar if (i) angles of one of them are congruent to the respective angles of the other, or (ii) the sides of one of them are proportional to the homologous sides of the other.

Arranging 2 similar triangles, so that the intercept theorem can be applied

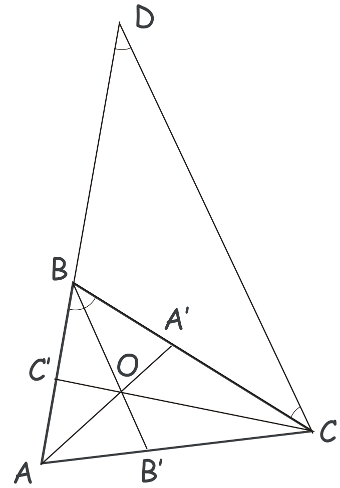
The similarity is closely related to the intercept (Thales) theorem. In fact this theorem is equivalent to the concept of similar triangles, i.e. it can be used to prove the properties of similar triangles, and similar triangles can be used to prove the intercept theorem. By matching identical angles one can always place 2 similar triangles in one another, obtaining the configuration in which the intercept theorem applies and vice versa the intercept theorem configuration always contains 2 similar triangles. In particular, a line parallel to any side of a given triangle cuts off a triangle similar to the given one.

Similarity tests for triangles.

- Two angles of one triangle are respectively congruent to the two angles of the other
- Two sides of one triangle are proportional to the respective two sides of the other, and the angles between these sides are congruent
- Three sides of one triangle are proportional to three sides of the other

Property of the bisector.

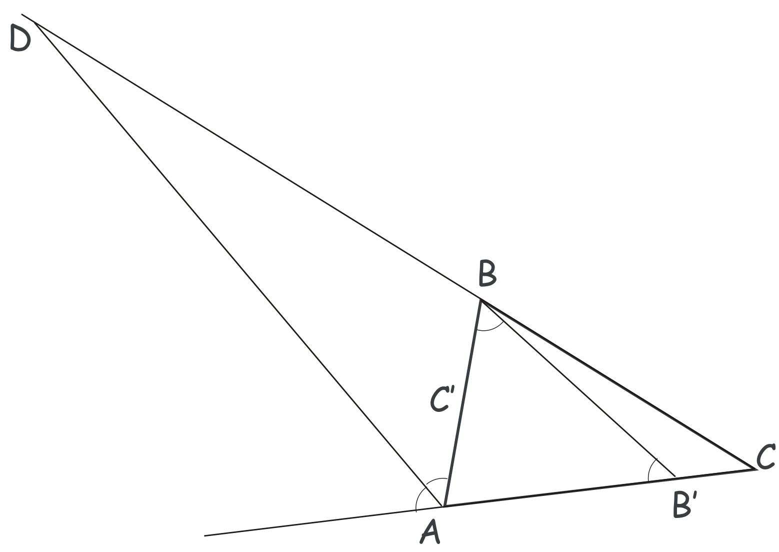
Theorem (property of the bisector). The bisector of any angle of a triangle divides the opposite side into parts proportional to the adjacent sides,



$$\frac{|AC'|}{|C'B|} = \frac{|AC|}{|BC|}, \frac{|BA'|}{|A'C|} = \frac{|AB|}{|AC|}, \frac{|CB'|}{|B'A|} = \frac{|BC|}{|AB|}$$

Proof. Consider the bisector BB’. Draw line parallel to BB’ from the vertex C, which intercepts the extension of the side AB at a point D. Angles B’BC and BCD have parallel sides and therefore are congruent. Similarly are congruent ABB’ and CDB. Hence, triangle CBD is isosceles, and |BD| = |BC|. Now, applying the intercept theorem to the triangles ABB’ and ACD, we obtain $\frac{|CB'|}{|B'A|} = \frac{|BD|}{|AB|} = \frac{|BC|}{|AB|}$.

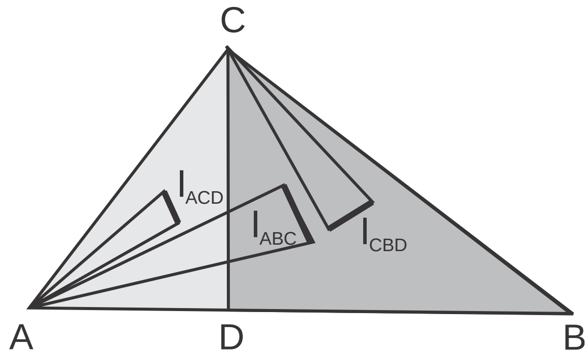
Theorem (property of the external bisector). The bisector of the exterior angle of a triangle intercepts the opposite side at a point (D in the Figure) such that the distances from this point to the vertices of the triangle belonging to the same line are proportional to the lateral sides of the triangle.



Proof. Draw line parallel to AD from the vertex B, which intercepts the side AC at a point B’. Angles ABB’ and DAB have parallel sides and therefore are congruent. Similarly, we see that angles AC’B and ABB’ are congruent, and, therefore, |AB’| = |AB|. Applying the intercept theorem, we obtain, $\frac{|DB|}{|DC|} = \frac{|AB'|}{|AC|} = \frac{|AB|}{|AC|}$.

Generalized Pythagorean Theorem.

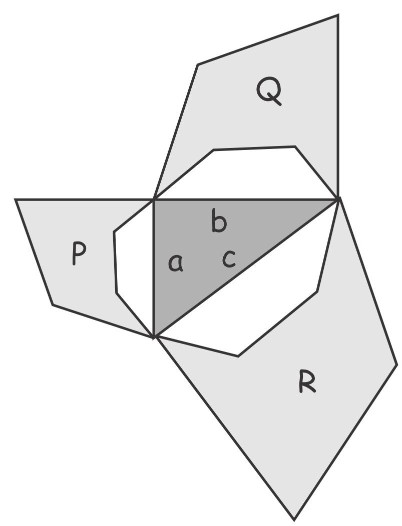
Theorem 1. For three homologous segments, l_{ABC}, l_{CBD} and l_{ACD} belonging to the similar right triangles ABC, CBD and ACD, where CD is the altitude of the triangle ABC drawn to its hypotenuse AB, the following holds,



$$l_{ACD}^2 + l_{CBD}^2 = l_{ABC}^2$$

Proof. If we square the similarity relation for the homologous segments, $\frac{l_{CBD}}{a} = \frac{l_{ACD}}{b} = \frac{l_{ABC}}{c}$, where $a = |BC|$, $b = |AC|$ and $c = |AB|$ are the legs and the hypotenuse of the triangle ABC, we obtain, $\frac{l_{CBD}^2}{a^2} = \frac{l_{ACD}^2}{b^2} = \frac{l_{ABC}^2}{c^2}$. Using the property of a proportion, we may then write, $\frac{l_{ACD}^2 + l_{CBD}^2}{a^2+b^2} = \frac{l_{ABC}^2}{c^2}$, wherefrom, by Pythagorean theorem for the right triangle ABC, $a^2 + b^2 = c^2$, we immediately obtain $l_{ACD}^2 + l_{CBD}^2 = l_{ABC}^2$.

Theorem 2. If three similar polygons, P, Q and R with areas S_P, S_Q and S_R are constructed on legs a, b and hypotenuse c, respectively, of a right triangle, then,



$$S_P + S_Q = S_R$$

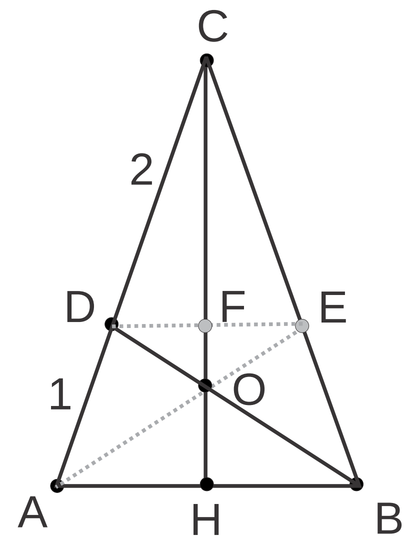
Proof. The areas of similar polygons on the sides of a right triangle satisfy $\frac{S_R}{S_P} = \frac{c^2}{a^2}$ and $\frac{S_R}{S_Q} = \frac{c^2}{b^2}$, or, $\frac{S_P}{a^2} = \frac{S_Q}{b^2} = \frac{S_R}{c^2}$. Using the property of a proportion, we may then write, $\frac{S_P+S_Q}{a^2+b^2} = \frac{S_R}{c^2}$, wherefrom, using the Pythagorean theorem for the right triangle, $a^2 + b^2 = c^2$, we immediately obtain $S_P + S_Q = S_R$.

Exercise. Show that for any proportion,

$$\left(\frac{a}{b}=\frac{c}{d}\right) \Rightarrow \left(\frac{a+c}{b+d}=\frac{a}{b}=\frac{c}{d}\right) \wedge \left(\frac{a-c}{b-d}=\frac{a}{b}=\frac{c}{d}, if\ b \neq d\right)$$

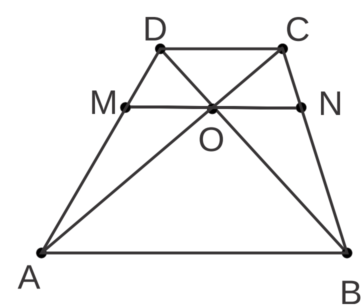
<u>Selected problems on similar triangles</u>.

Problem 1 (homework problem #4). In the isosceles triangle ABC point D divides the side AC into segments such that $|AD|:|CD| = 1:2$. If CH is the altitude of the triangle and point O is the intersection of CH and BD, find the ratio $|OH|$ to $|CH|$.



Solution. First, let us perform a supplementary construction by drawing the segment DE parallel to AB, $DE||AB$, where point E belongs to the side CB, and point F to DE and the altitude CH. Notice the similar triangles, $AOH \sim DOF$, which implies, $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|}$. By Thales theorem, $\frac{|AH|}{|DF|} = \frac{|AC|}{|AD|} = 1 + \frac{|CD|}{|AD|} = \frac{3}{2}$, and $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|} = \frac{2}{3}$, so that $\frac{|FH|}{|OH|} = \frac{|FO|+|OH|}{|OH|} = \frac{5}{3}$. $\frac{|CH|}{|OH|} = \frac{|CH|}{|FH|}\frac{|FH|}{|OH|} = 3 \cdot \frac{5}{3} = 5$, because $\frac{|CH|}{|FH|} = 1 + \frac{|CF|}{|FH|} = 1 + \frac{|CD|}{|DA|}$. Therefore, the sought ratio is, $\frac{|OH|}{|CH|} = \frac{1}{5}$.

Problem 2 (homework problem #5). In a trapezoid $ABCD$ with the bases $|AB| = a$ and $|CD| = b$, segment MNparallel to the bases, $MN||AB$, connects the opposing sides, $M \in [AD]$ and $N \in [BC]$. MN also passes through the intersection point O of the diagonals, AC and BD, as shown in the Figure. Prove that $|MN| = \frac{2ab}{a+b}$.



Solution. By Thales theorem applied to vertical angles AOB and DOC and parallel lines AB and CD, $\frac{|AM|}{|MD|} = \frac{|BN|}{|NC|} = \frac{|AB|}{|DC|} = \frac{a}{b}$. Consequently, $\frac{|AD|}{|MD|} = \frac{|AM|+|MD|}{|MD|} = \frac{a}{b} + 1 = \frac{|BN|+|NC|}{|NC|} = \frac{|BC|}{|NC|}$. Now, applying the same Thales theorem to angles ADB and ACB and parallel lines MN and AB, we obtain, $\frac{|MO|}{|AB|} = \frac{|MD|}{|AD|} = \frac{1}{\frac{a}{b}+1}$ and $\frac{|ON|}{|AB|} = \frac{|NC|}{|BC|} = \frac{1}{\frac{a}{b}+1}$. Hence, $\frac{|MO|}{|AB|} + \frac{|ON|}{|AB|} = \frac{|MN|}{|AB|} = \frac{2}{\frac{a}{b}+1}$, and $|MN| = \frac{2ab}{a+b}$.