MATH 7
ASSIGNMENT 16: VECTORS IN THE PLANE
MAR 13, 2022

Vectors

A **vector** is a directed segment. We denote the vector from A to B by $\overrightarrow{AB}$. We will also frequently use lower-case letters for vectors: $\vec{v}$.

We will consider two vectors to be the same if they have the same length and direction; this happens exactly when these two vectors form two opposite sides of a parallelogram. Using this, we can write any vector $\vec{v}$ as a vector with tail at given point A. We will sometimes write $A + \vec{v}$ for the head of such a vector.

Vectors are used in many places. For example, many physical quantities(velocities, forces, etc) are naturally described by vectors.

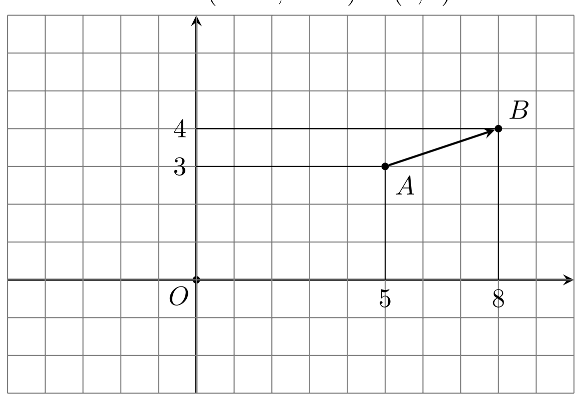
Vectors in coordinates

Recall that every point in the plane can be described by a pair of numbers – its coordinates. Similarly, any vector can be described by two numbers, its x-coordinate and y-coordinate: for a vector $\overrightarrow{AB}$, with tail $A = (x_1, y_1)$ and head $B = (x_2, y_2)$, its coordinates are

$$\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$$

For example, on picture below,

$$\overrightarrow{AB} = (8 - 5, 4 - 3) = (3, 1)$$



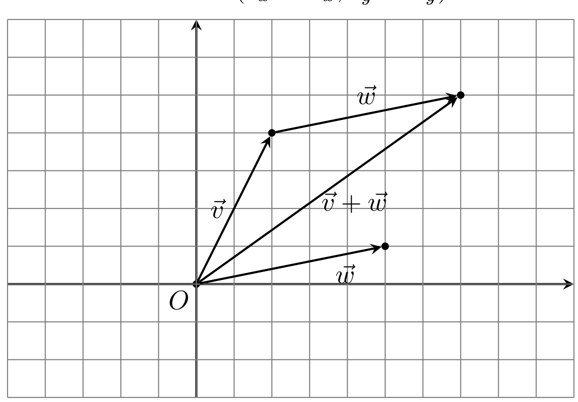
Operations with vectors

Let $\vec{v}$, $\vec{w}$ be two vectors. Then we define a new vector, $\vec{v} + \vec{w}$ as follows: choose A, B, C so that $\vec{v} = \overrightarrow{AB}$, $\vec{w} = \overrightarrow{BC}$; then define

$$\vec{v} + \vec{w} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

In coordinates, it looks very simple: if $\vec{v} = (v_x, v_y)$, $\vec{w} = (w_x, w_y)$, then

$$\vec{v} + \vec{w} = (v_x + w_x, v_y + w_y)$$



Theorem. *So defined addition is commutative and associative:*

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$
$$(\vec{v_1} + \vec{v_2}) + \vec{v_3} = \vec{v_1} + (\vec{v_2} + \vec{v_3})$$

There is no obvious way of multiplying two vectors; however, one can multiply a vector by a number: if $\vec{v} = (v_x, v_y)$ and t is a real number, then we define

$$t\vec{v} = (tv_x, tv_y)$$

Again, we have the usual distributive properties.

Homework

1. (a) Let $A = (3, 6)$, $B = (5, 2)$. Find the coordinates of the vector $\vec{v} = \overrightarrow{AB}$ and coordinates of the points $A + 2\vec{v}$; $A + \frac{1}{2}\vec{v}$; $A - \vec{v}$.
(b) Repeat part (a) for points $A = (x_1, y_1)$,$B = (x_2, y_2)$

2. Let $A = (x_1, y_1)$,$B = (x_2, y_2)$. Show that the midpoint M of segment AB has coordinates $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ and that $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$.
[Hint: point M is $A + \frac{1}{2}\vec{v}$, where $\vec{v} = \overrightarrow{AB}$].

3. Let AB be a segment, and M -– a point on the segment which divides it in the proportion 2:1, i.e., $|AM| = 2|MB|$. Let O be the origin. Show that $\overrightarrow{OM} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB} = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OB}$

4. Consider a parallelogram $ABCD$ with vertices $A(0, 0)$, $B(3, 6)$, $D(5, -2)$. Find the coordinates of:
(a) vertex C
(b) midpoint of segment BD
(c) Midpoint of segment AC

5. Repeat the previous problem if coordinates of B are (x_1, y_1), and coordinates of D are (x_2, y_2). Use the result to prove that diagonals of a parallelogram bisect each other (i.e., the intersection point is the midpoint of each of them).

Additional Problems (Optional)

1. Consider triangle $\triangle ABC$ with $A(2, 1)$, $B(3, 8)$, $C(7, 0)$.
(a) Find the coordinates of the midpoints A_1 of segment BC; of midpoint B_1 of segment AC; of midpoint C_1 of segment AB.
(b) Find the coordinates of the point on the median AA_1 which divides AA_1 in proportion $2 : 1$ (see problem 3). Repeat the same for two other medians BB_1 and CC_1.

2. Let A_1 and B_1 be the midpoints of the sides BC and AC of $\triangle ABC$. Prove that
(a) $\overrightarrow{AA_1} = 2(\overrightarrow{AB} + \overrightarrow{AC})$
(b) $\overrightarrow{A_1B_1} = \frac{1}{2}\overrightarrow{AB}$

3. Let A_1 and B_1 be the midpoints of the sides BC and AD of quadrilateral $ABCD$. Prove that
(a) $\overrightarrow{AA_1} = 2(\overrightarrow{AB} + \overrightarrow{AC})$
(b) $\overrightarrow{A_1B_1} = \frac{1}{2}\overrightarrow{AB}$