MATH 10
ASSIGNMENT 16: COMPLETENESS AXIOM
FEB 13, 2022

Least upper bound

Definition. A number M is called an *upper bound* of set S if for any $s \in S$, we have $s \leq M$.

A number M is called the *least upper bound* of set S (notation: $M = \sup(S)$) if

1. M is an upper bound of S, i.e. $\forall s \in S: \; s \leq M$
2. M is the smallest possible upper bound: if $M' < M$, then there exists $s \in S$ such that $s > M'$ and thus M' cannot be an upper bound.

Note that it is possible that the least upper bound is not in S.

Condition (2) can be rewritten in this form:

$$\forall \varepsilon > 0: \; \exists s \in S: \; M - \varepsilon < s \leq M$$

which is sometimes more convenient.

Axiom (Completeness axiom)**.** *For any set $S \subset \mathbb{R}$ which is bounded above, there exists the least upper bound.*

This is one of the defining properties of real numbers. There are many equivalent formulations of this property, such as Theorem 1 below or nested intervals property (see Problem 4). It is taken as an axiom.

Note that this property fails for rational numbers: for example, set $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ is bounded above but has no least upper bound (in $\mathbb{Q}$). It does have a least upper bound in $\mathbb{R}$, namely $\sqrt{2}$.

Limits of bounded sequences

Theorem 1. *Any increasing bounded sequence has a limit.*

Problems

1. Compute the limits of the following sequences (if they exist):
   (a) $\lim \dfrac{n^3 + 5n - 7}{(50n + 3)(2n - 7)}$
   (b) $\lim \frac{(-1)^n}{2^n}$
2. Find the least upper bound of the following sets (if they exist):
   (a) $S = [0, 1]$
   (b) $S = (0, 1)$
   (c) $\{1 - \frac{1}{n}\}$, $n = 1, 2, \dots$
   (d) $\{x \in \mathbb{R} \mid x^2 < 2\}$
3. Prove Theorem 1. [Hint: let $M = \sup\{a_n\}$. Show that then M is the limit.]
4. (a) Consider a sequence of nested intervals:
   $$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \dots$$
   Use completeness axiom to prove that then, there exists a point c which belongs to all of these intervals: for all n, $a_n \leq c \leq b_n$. Is such a point unique?
   [Hint: any of the b_i is an upper bound of set $S = \{a_1, \dots, a_k, \dots\}$.]
   (b) Show that the statement of the previous part fails if we replace closed intervals by open intervals (a_n, b_n). [Hint: consider intervals $(0, \frac{1}{n})$.]
5. Let
   $$a_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n \cdot (n+1)}$$
   (a) Compute a_1, a_2, a_3, a_4. Can you guess a general formula? [Hint: $\frac{1}{n \cdot (n+1)} = \frac{1}{n} - \frac{1}{n+1}$.]
   (b) Find $\lim a_n$

(c) Let now

$$b_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

Use inequality $\frac{1}{(n+1)^2} \leq \frac{1}{n\cdot(n+1)}$ to prove that $b_n \leq a_{n-1} + 1$

(d) Prove that b_n has a limit. [This limit is actually equal to $\pi^2/6$, but it is rather hard to prove.]