

Classwork 13.
Algebra.



Rational number is a number which can be represented as a ratio of two integers:

$$a = \frac{p}{q}; \quad p \in Z, \text{ and } q \in N, \quad (Z = \{\pm \dots, \pm 1, 0\}, N = \{1, 2, \dots\})$$

Rational numbers can be represented as infinite periodical decimals (in the case of denominators containing only powers of 2 and 5 the periodical bloc of such decimal is 0).

Numbers, which can't be express as a ratio (fraction) $\frac{p}{q}$ for any integers p and q are irrational numbers. Their decimal expansion is not finite, and not periodical.

Examples:

0.01001000100001000001...

0.123456789101112131415161718192021...

What side the square with the area of $a \text{ m}^2$ does have? To solve this problem, we have to find the number, which gives us a as its square. In other words, we have to solve the equation

$$x^2 = a$$

This equation can be solved (has a real number solution) only if a is nonnegative ($a \geq 0$) number. It can be seen very easily;

If $x = 0$, $x \cdot x = x^2 = a = 0$,

If $x > 0$, $x \cdot x = x^2 = a > 0$,

If $x < 0$, $x \cdot x = x^2 = a > 0$,

We can see that the square of any real number is a nonnegative number, or there is no such real number that has negative square.

Square root of a (real nonnegative) number a is a number, square of which is equal to a .

There are only 2 square roots from any positive number, they are equal by absolute value, but have opposite signs. The square root from 0 is 0, there is no any real square root from negative real number.

Examples:

1. Find square roots of 16: 4 and (-4) , $4^2 = (-4)^2 = 16$
2. Numbers $\frac{1}{7}$ and $(-\frac{1}{7})$ are square roots of $\frac{1}{49}$, because $\frac{1}{7} \cdot \frac{1}{7} = (-\frac{1}{7}) \cdot (-\frac{1}{7}) = \frac{1}{49}$
3. Numbers $\frac{5}{3}$ and $(-\frac{5}{3})$ are square roots of $\frac{25}{9}$, because

$$\left(\frac{5}{3}\right)^2 = \frac{5}{3} \cdot \frac{5}{3} = \left(-\frac{5}{3}\right)^2 = \left(-\frac{5}{3}\right) \cdot \left(-\frac{5}{3}\right) = \frac{25}{9}$$

Principal square root of a (real nonnegative) number a is a nonnegative number, square of which is equal to a .

There is a special sign for the principal square root of a number a : \sqrt{a} .

Examples;

1. $\sqrt{25} = 5$, it means that principal square root of 25 is 5, as a nonnegative number, square of which is 25. Square roots of 25 are 5 and (-5) , or $\pm\sqrt{25} = \pm 5$
2. Square roots of 121 are 11 and (-11) , or $\pm\sqrt{121} = \pm 11$
3. Square roots of 2 are $\pm\sqrt{2}$.
4. A few more:

$$\begin{array}{lllll} \sqrt{0} = 0; & \sqrt{1} = 1; & \sqrt{4} = 2; & \sqrt{9} = 3; & \sqrt{16} = 4; \\ \sqrt{25} = 5; & \sqrt{\frac{1}{64}} = \frac{1}{8}; & \sqrt{\frac{36}{25}} = \frac{6}{5} \end{array}$$

Base on the definition of arithmetic square root we can right

$$(\sqrt{a})^2 = a$$

To keep our system of exponent properties consistent let's try to substitute $\sqrt{a} = a^k$. Therefore,

$$(\sqrt{a})^2 = (a^k)^2 = a^1$$

But we know that

$$(a^k)^2 = a^{2k} = a^1 \Rightarrow 2k = 1, k = \frac{1}{2}$$

And we can agree to consider arithmetic square root as fractional exponent

$$\sqrt{a} = a^{\frac{1}{2}}$$

$$(\sqrt{a})^2 = a, \quad (\sqrt{b})^2 = b, \quad (\sqrt{a})^2 (\sqrt{b})^2 = (\sqrt{a}\sqrt{b})^2 = ab = (\sqrt{ab})^2 \Rightarrow \sqrt{a}\sqrt{b} = \sqrt{ab}$$

To solve equation $x^2 = 23$ we have to find two sq. root of 23. $x = \pm\sqrt{23}$. 23 is not a perfect square as 4, 9, 16, 25, 36 ...

Geometry.

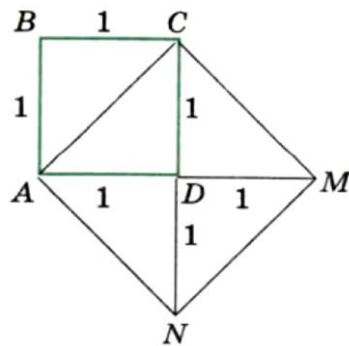
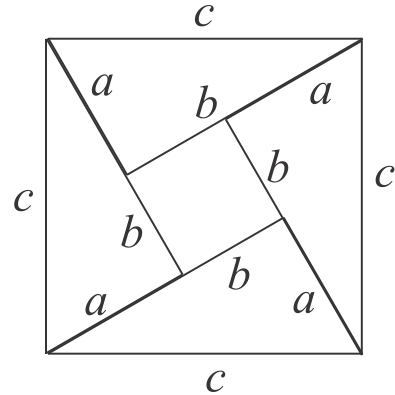
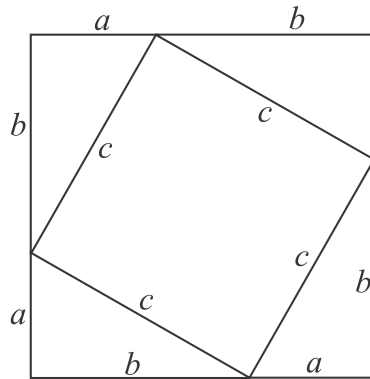
Pythagorean theorem.

4 identical right triangles are arranged as shown on the picture. The area of the big square is $S = (a + b) \cdot (a + b) = (a + b)^2$, the area of the small square is $s = c^2$. The area of 4 triangles is $4 \cdot \frac{1}{2}ab = 2ab$. But also can be represented as $S - s = 2ab$

$$2ab = (a + b) \cdot (a + b) - c^2 = a^2 + 2ab + b^2 - c^2$$

$$\Rightarrow a^2 + b^2 = c^2$$

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The length of the segment [AC] is $\sqrt{2}$ (from Pythagorean theorem). The area of the square ACMN is twice the area of the square ABCD. Let assume that the $\sqrt{2}$ is a rational number, so it can be represented as a ratio $\frac{p}{q}$, where $\frac{p}{q}$ is nonreducible fraction.

$$\left(\frac{p}{q}\right)^2 = 2 = \frac{p^2}{q^2}$$

Or $p^2 = 2q^2$, therefore p^2 is an even number, and p itself is an even number, and can be represented as $p = 2p_1$, consequently

$$p^2 = (2p_1)^2 = 4p_1^2 = 2q^2$$

$2p_1^2 = q^2 \Rightarrow q$ also is an even number and can be written as $q = 2q_1$.

$$\frac{p}{q} = \frac{2p_1}{2q_1}$$

therefore fraction $\frac{p}{q}$ can be reduced, which is contradict the assumption. We proved that the $\sqrt{2}$ isn't a rational number by contradiction.

Exercises:

1. Prove that the value of the following expressions is a rational number.

a. $(\sqrt{2} - 1)(\sqrt{2} + 1)$

b. $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$

c. $(\sqrt{2} + 1)^2 + (\sqrt{2} - 1)^2$

d. $(\sqrt{7} - 1)^2 + (\sqrt{7} + 1)^2$

e. $(\sqrt{7} - 2)^2 + 4\sqrt{7}$

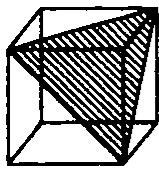
2. Simplify the expressions:

a. $\frac{a^{-2} - b^{-2}}{a^{-1} + b^{-1}};$

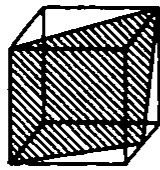
b. $\frac{a^{-4} - b^{-4}}{a^{-2} + b^{-2}}$

c. $\frac{\frac{2a}{1-a}}{1 - \left(\frac{1-a}{2a}\right)^{-1}}$

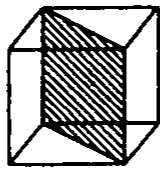
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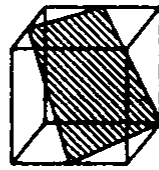
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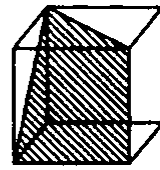
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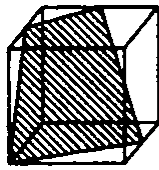
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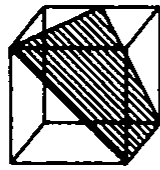
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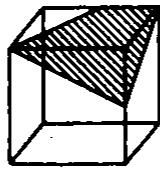
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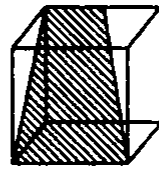
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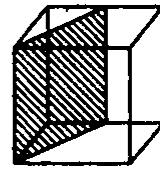
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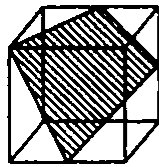
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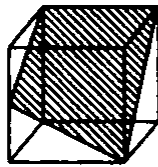
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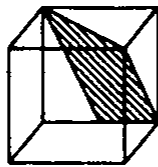
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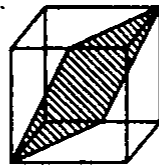
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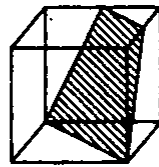
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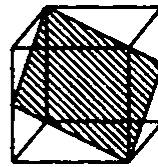
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