Classwork 4.

Algebra.

Properties of exponent.

Let us explore the definition of the exponent, a^m, when m is a positive integer. Then

$$a^n = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n\ times} \tag{1}$$

Based on this definition (1) we can show that

$$a^n \cdot a^m = \underbrace{a \cdot a \ldots \cdot a}_{n\ times} \cdot \underbrace{a \cdot a \ldots \cdot a}_{m\ times} = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n+m\ times} = a^{n+m} \tag{2}$$

and

$$(a^n)^m = \underbrace{a^n \cdot a^n \cdot \ldots \cdot a^n}_{m\ times} = \underbrace{\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times} \cdot \ldots \cdot \underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}}_{m\ times} = a^{n \cdot m} \tag{3}$$

If we want to multiply $a^n = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n\ times}$ by another a we will get the following expression:

$$a^n \cdot a = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n\ times} \cdot a = \underbrace{a \cdot a \cdot a \cdot a \ldots \cdot a}_{n+1\ times} = a^{n+1} = a^n \cdot a^1 \tag{4}$$

In order to have the set of power properties consistent, $a^1 = a$ for any number a.

We can multiply any number by 1, this operation will not change the number, so if

$$a^n = a^n \cdot 1 = a^{n+0} = a^n \cdot a^0 \tag{5}$$

In order to have the set of properties of exponent consistent, $a^0 = 1$ for any number a.

$$\frac{a^n}{a^m} = \frac{\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}}, n > m$$

$$\frac{a^n}{a^m} = \frac{\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}} = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n-m\ times} = a^{n-m} \tag{6}$$

We can rewrite the expression (6) as

$$\frac{a^n}{a^m} = \frac{\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}} = \left(\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}\right) : \left(\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}\right) = a^n : a^m = a^{n-m} \tag{7}$$

We see that based on the definition of the exponentiation *n*-th and *m*-th powers add up if a^n and a^m are multiplied and subtract if a^n is divided by a^m. We know, on the other hand, that division is a multiplication by inverse number.

Let rewrite (6) again

$$\frac{a^n}{a^m} = a^n : a^m = a^{n+(-m)} = a^n \cdot \frac{1}{a^m} = a^n \cdot a^{-m} \ , a \neq 0 \ \text{for any } n \text{ and } \ m \ \in \mathbb{N}$$

Negative power of a number, not equal to 0, can be defined as

$$a^{-m} = \frac{1}{a^m} \ , \ \ a \neq 0$$

So, all properties of the exponentiation can be defined as:

1. $a^n = \underbrace{a \cdot a \cdot a \dots \cdot a}_{n\ times}$
2. $a^n \cdot a^m = a^{n+m}$
3. $(a^n)^m = a^{n \cdot m}$
4. $a^1 = a$, for any a
5. $a^0 = 1$, for any a
6. $\frac{1}{a^m} = a^{-m}$, for any $a \neq 0$
7. $(a \cdot b)^n = a^n \cdot b^n$

Also, if there are two numbers a and b:

$$(a \cdot b)^n = \underbrace{(a \cdot b) \cdot \dots \cdot (a \cdot b)}_{n\ times} = \underbrace{a \cdot \dots \cdot a}_{n\ times} \cdot \underbrace{b \cdot \dots \cdot b}_{n\ times}$$
$$= a^n \cdot b^n \qquad (9)$$

Last property is

$$(a \cdot b)^n = a^n \cdot b^n$$

I. A positive number raised into any power will result a positive number.

- Because the product of any number of positive factors gives as an outcome a positive number, and exponentiation is a product of same factors, we proved the statement.

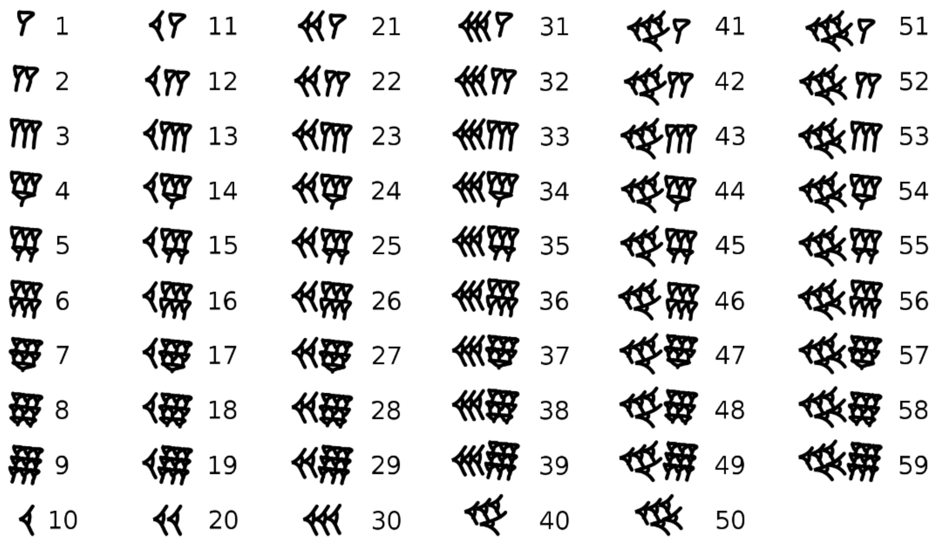
II. A negative number, raised in a power, represented by an even number is positive, represented by an odd number is negative.

- Even number m of negative factors produces $\frac{m}{2}$ pairs of multiplied negative numbers each of which is positive, as we know, of product any quantity of positive factors is positive.
- For an odd number n, $\frac{n-1}{2}$ pairs of negative factors will be multiplying by one more negative number, so the whole product will be negative.

Numeral systems.

Over the long centuries of human history, many different numeral systems have appeared in different cultures. The oldest systems weren't a place-valued system, but sometimes use the position of the "digit" to show units. For example, the Babylonian system (form about 2000 BC) used only two symbols to write any number between 1 and 60:

 to count units and  to count tens.



By Josell7 - File:Babylonian_numerals.jpg, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=9862983

Number 62 was shown  as which means one time 60 and 2. The use of this sexagesimal (60-based) system is still visible, we have 60 minutes in one hour, 60 seconds in a minute, 360° for the full turn around.

Another very well known numeral system is roman, it was used for thousands of years and in some cases is still used now as well. It's a “decimal”, 10 based system, but the symbols (letters) are used in an unusual way.

Symbol					C	D	M
Value					100	500	1000

For example, 4 is one less than 5, so 4 can be written as IV. Same principle of subtractive notation is used for 9 -> IX, 40 and 90 -> XL and XC, 400 and 900 -> CD and CM

1	2	3	4	5	6	7	8	9	10
I	II	III	IV	V	VI	VII	VIII	IX	X

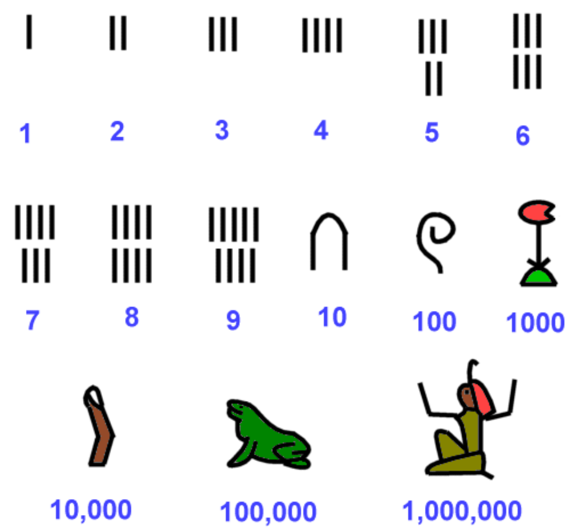
Some other examples:

- 29 = XX + IX = **XXXIX**.
- 347 = CCC + XL + VII = **CCCXLVI**.
- 789 = DCC + LXXX + IX = **DCCLXXXIX**.

- 2,421 = MM + CD + XX + I = **MMCDXXI**.

  Any missing place (represented by a zero in the place-value equivalent) is omitted, as in Latin (and English) speech:

- 160 = C + LX = **CLX**
- 207 = CC + VII = **CCVII**
- 1,009 = M + IX = **MIX**
- 1,066 = M + LX + VI = **MLXVI**



Egyptian numeral system was decimal (10-based), similar to the one we use now, but also wasn't placed-valued and didn't use the position of the symbol in any way.

To write a number, you need to draw the symbol of units, tens, and so on as many times as there are units, tens, and so on. For example, the number 4622 should be written as (order in which all the symbols are written is not important).:

The most familiar numeral system for us is our decimal place-valued system, were the position of the digit defines its value. In the extended form we can right any number as a sum of 10^n multiplied by a numbers.

$$2345 = 2000 + 300 + 40 + 5 = 1000 \cdot 2 + 100 \cdot 3 + 10 \cdot 4 + 5 = 10^3 \cdot 2 + 10^2 \cdot 3 + 10^1 \cdot 4 + 10^0 \cdot 5$$

This system can be extended to fractional parts of the number:

$$2345.23 = 10^3 \cdot 2 + 10^2 \cdot 3 + 10^1 \cdot 4 + 10^0 \cdot 5 + \frac{1}{10} \cdot 2 + \frac{1}{100} = \\ = 10^3 \cdot 2 + 10^2 \cdot 3 + 10^1 \cdot 4 + 10^0 \cdot 5 + 10^{-1} \cdot 2 + 10^{-2} \cdot 3$$

Can non-decimal place-value system be created? For example, with base 5?

Let see, how we can create this kind of system (we use our normal digits).

Num_{10}	1	2	3	4	5	6	7	8	9	10
Num_5	1	2	3	4	10	11	12	13	14	20

11	12	13	14	15	16	17	18	19	20
21	22	23	24	30	31	32	33	34	40

21	22	23	24	25	26	27	28	29	30
41	42	43	44	100	101	102	103	104	110

We only have 5 digits (0, 1, 2, 3, 4), and 4 first “natural” numbers in such system will be represented just with digits. Number 5 then should be shown as a 2-digit number, with fist digit 1 (place – value equal to 5^1) and 0 of “units”. Any number is now written if the form

$5^3 \cdot m + 5^2 \cdot n + 5^1 \cdot k + 5^0 \cdot p$

$33 = 25 + 5 + 3 = 5^2 \cdot 1 + 5^1 \cdot 1 + 3 \rightarrow 113_5$

$195 = 125 + 2 \cdot 25 + 20 = 5^3 + 5^2 \cdot 2 + 5^1 \cdot 4 + 0 \rightarrow 1240_5$

And vice versa, if we need transform the number form 5-base to decimal system:

$2312_5 \rightarrow 5^3 \cdot 2 + 5^2 \cdot 3 + 5^1 \cdot 1 + 5^0 \cdot 2 = 250 + 75 + 5 + 2 = 332$

Let’s try to introduce a new digit ├ for 10 and then create an 11 based system.

$11^2 = 121,\qquad 11^3 = 1331$

$890 = 847 + 33 + 10 = 121 \cdot 7 + 11 \cdot 3 + 10 = 11^2 \cdot 7 + 11^1 \cdot 3 + 11^0 \cdot 10 \rightarrow 73{\vdash}_{11}$

$4{\vdash}2_{11} \rightarrow 11^2 \cdot 4 + 11^1 \cdot 10 + 11^0 \cdot 2 = 484 + 110 + 2 = 596$

There is another very important place-value system: binary system, base 2 system where only two digits exist; 0, and 1.

Num_{10}	1	2	3	4	5	6	7	8	9	10
Num_2	1	10	11	100	101	110	111	1000	1001	1010

In this system each number is represented as

$2^3 \cdot (0,1) + 2^2 \cdot (0,1) + 2^1 \cdot (0,1) + 2^0 \cdot (0,1)$

$11 = 8 + 2 + 1 = 2^3 \cdot 1 + 2^2 \cdot 0 + 2^1 \cdot 1 + 2^0 \cdot 1 \rightarrow 1011_2$

$75 = 64 + 8 + 2 + 1 = 2^6 + 2^3 + 2 + 1 = 2^6 \cdot 1 + 2^5 \cdot 0 + 2^4 \cdot 0 + 2^3 \cdot 1 + 2^2 \cdot 0 + 2^1 \cdot 1 + 2^0 \cdot 1 \rightarrow 1001011_2$

Exercises:

1. Compute:
   a. $(-3)^3$;
   b. -3^3;
   c. 2^7;
   d. $(-2)^7$;
   e. -2^7;
   f. $(2 \cdot 3)^3$;
   g. $2 \cdot 3^3$;
   h. $\left(\frac{1}{3}\right)^2$;
   i. $\frac{1}{3^2}$;
   j. 3^{-2};
   k. $(-3)^{-2}$;
   l. $(-5 \cdot 2)^3$

   Remember, that $a^n : a^m = a^{n-m} = a^{n+(-m)} = a^n \cdot \frac{1}{a^m} = a^n \cdot a^{-m}$

2. Write the following expressions in a shorter way replacing product with power:

   Examples:

   $(-a) \cdot (-a) \cdot (-a) \cdot (-a) = (-a)^4$, $\quad 3m \cdot m \cdot m \cdot 2k \cdot k \cdot k \cdot k = 6m^3k^4$

   $(-y) \cdot (-y) \cdot (-y) \cdot (-y)$; $\qquad (-5m)(-5m) \cdot 2n \cdot 2n \cdot 2n$;

$-y \cdot y \cdot y \cdot y;$ $\quad$ $-5m \cdot m \cdot 2n \cdot n \cdot n;$

$(ab) \cdot (ab) \cdot (ab) \cdot (ab) \cdot (ab) \cdot (ab);$ $\quad$ $p - q \cdot q \cdot q \cdot q \cdot q;$

$a \cdot b \cdot b \cdot b \cdot b \cdot b;$ $\quad$ $(p-q) \cdot (p-q) \cdot (p-q) \cdot (p-q);$

3. Write the following expressions replacing exponent with a product of several factors:

   Examples: $(-x)^3 = (-x) \cdot (-x) \cdot (-x);\quad 3y - a^4 = 3y - a \cdot a \cdot a \cdot a$

   $(-n)^3;$ $\quad$ $(-mn)^4$ $\quad$ $2x - y^3$

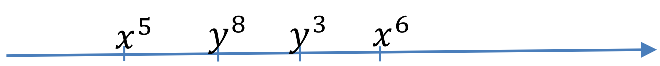
   $-x^2;$ $\quad$ $-mn^4$ $\quad$ $(a+3b)^2$

   $(2c)^2;$ $\quad$ $2c^2;$ $\quad$ $(2x-y)^3$

4.

$$x^5 < y^8 < y^3 < x^6$$

Where 0 should be placed?



5. Evaluate:

   $a.\ \ 8 + 7^2;\quad (8+7)^2;\quad 8^2 + 7^2;$
   $b.\ \ (11-6)^3;\quad 11 - 6^3;\quad 11^3 - 6^3;$
   $c.\ \ 5 \cdot 2^4;\quad (5 \cdot 2)^4;\quad 5^4 \cdot 2^4;$
   $d.\ \ (12:2)^3;\quad 12:2^3;\quad 12^3:2^3$

6. Sum of two natural numberы is 45. First number will give the remainder 4 upon division by 12, second number will give the remainder 5 upon division by 12. What are these numbers?

7. Represent a^{24} as an exponent with the base

   $a.\ a^2;\quad b.\ a^3;\quad c.\ a^4;\quad d.\ a^6;\quad e.\ a^8;\quad f.\ a^{12}$

8. Compare the following exponents:

   $a.\ \ 2^{10}\ \ and\ \ 10^3;\qquad b.\ \ 10^{100}\ \ and\ 100^{10}$

$c.\ \ 2^{300} \quad and\ \ 200;$ $\quad d.\ \ 31^{16} \quad and\ 17^{20};$ $\quad e.\ \ 4^{53} \quad and\ 15^{45}$

9. Prove that
   $8^5 + 2^{11}$ is divisible by 17
   $9^7 - 3^{10}$ is divisible by 20

10. A farmer has a cow, a goat and a goose. The cow and the goat will eat all the grass on his meadow in 45 days, the cow and the goose will eat all the grass on the same meadow in 60 days, and the goat and the goose will eat all the grass on the meadow in 90 days. How many days will it take them altogether to eat all the grass on the meadow? (we assume that the new grass is not growing.)
11. Write numbers 45 and 165 in binary system

12. Write the numbers, written in the binary system in decimal system:
    $a.\ \ 11011011;$ $\quad b. 10001101,$ $\quad c.\ \ 111111111$

13. Write the numbers 245 and 324 in 6-based place-value system. Remember, that in this system you will have only 0, 1, 2, 3. 4, and 5 as digits.

14. Write the numbers 234_6 and 403_6 written in the 6-based place-value system (small number 6 shows that the number is not in decimal, but in 6-based system) in decimal system.

15. How to arrange 127 dollar bills in seven wallets so that any amount from 1 to 127 dollars could be issued without opening the wallets?

16. Robert thought of a number not less than 1 and not more than 1000. Julia is allowed to ask only such questions to which Robert can answer “yes” or “no” (Robert always tells the truth). Can Julia determine the hidden number in 10 questions?

17. There is a bag of sugar, a scale and a weight of 1 g. Is it possible to measure 1 kg of sugar in 10 weights?