Class work 1. Algebra.

Algebra.

1. Natural numbers.

According to the modern studies, many animals' species have a sense of amount and quantity. In many different experiments animals have shown the ability to differentiate between smaller and bigger amount of food, quantities of things such as 1, 2, and 3. In captivity, after training, they can show even better results. Prehistoric people have introduced the special words to indicate the number of items in a group (number of elements in a set, as we are saying now). We can even assume that at the beginning, different words were used to specify the same number of the different objects. Only after thousands of years the abstract concept of "numbers" was separated from the number of real objects in a group. That moment can be considered as a beginning of mathematics.

2. Properties of the arithmetic operations.

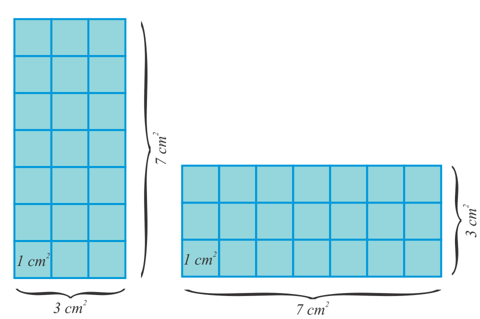
Addition		Multiplication
$a+b=b+a$	← Commutative property →	$a \times b=b \times a$
$a+(b+c)=(a+b)+c$	← Associative property →	$a \times (b \times c)=(a \times b) \times c$

Distributive property ↓

$$a \times (b+c)=a \times b + a \times c$$

$$a \times (b-c)=a \times b - a \times c$$

Commutative and associative properties of addition are easy to understand. Multiplication is just a shorter way to write the addition of equal groups, so commutative and associative properties of multiplication can be visualized and understood with the help of the rectangle area. (See the picture). Areas of identical rectangles are equal,



$$S = 3cm^2 \cdot 7 = 7cm^2 \cdot 3 = 3cm \cdot 7cm = 21cm^2$$

The distributive property can be explained with the definition of multiplication as well;

$2 \cdot (3+7) = (3+7) + (3+7) = 3 + 3 + 7 + 7 = 2 \cdot 3 + 2 \cdot 7$ and it is true for any numbers.

Or, in a general way:

$$a(b+c) = \underbrace{(b+c) + (b+c) + \cdots + (b+c)}_{a\ times} = \underbrace{b + b + \cdots + b}_{a\ times} + \underbrace{c + c + \cdots + c}_{a\ times} = ab + ac$$

In the expression above a and $(b+c)$ are factors, a and b and a and c also factors of products of ab and ac. In the sum of $ab + ac$ a is a common factor, and can be factor out:

$$ab + ac = \underbrace{b + b + \cdots + b}_{a\ times} + \underbrace{c + c + \cdots + c}_{a\ times} = \underbrace{(b+c) + (b+c) + \cdots + (b+c)}_{a\ times} = a(b+c)$$

Farmer put green and red grapes into boxes. Each box contains 5lb of grapes. How many pounds of green and red grapes altogether did farmer put into boxes if he had 10 boxes of green and 8 boxes of red grapes? Is there any difference between 2 following expressions?

$$5 \cdot (10 + 8) \text{ or } 5 \cdot 10 + 5 \cdot 8$$

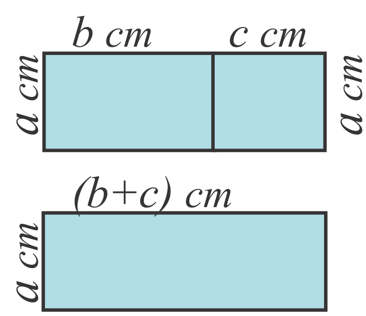
What is represented by the first expression? By the second?

Another example:

Combined area of two rectangles S_2 equals to $a \cdot b + a \cdot c$, and the area of one big rectangle is $S_1 = a(b+c)$:

$$S_1 = a(b+c) = a \cdot b + a \cdot c = S_2$$

(see the picture on the right).



3. Divisibility.

We say that a natural number is divisible by another natural number if the result of this operation is a natural number. If this is not the case then we can divide a number with a remainder.

If *a* and *n* are natural numbers, the result of a division operation of *a* ÷ *n* will be a quotient *c*, such that

$$a = b \times c + r$$

Where *r* is a remainder of a division *a* ÷ *b*. If *r* is 0, then we can tell that *a* is divisible by *b*.

- If we want to divide *m* by 15, what numbers we can get as a remainder?

If the remainder is 0, then quotient and divisor are both factors of dividend, $a = b \cdot c$, and to divide a number a by another number, $\boldsymbol{b}$, means to find such number $\boldsymbol{c}$, that $c \cdot b$ will give us a. So, because the product of 0 and any number is 0, then there is no such arithmetic operation as division by 0.

4. Divisibility rules.

A statement (or proposition) is a sentence that is either true or false, but not both. So '3 is an odd integer' is a statement.
But 'π is a cool number' is not a (mathematical) statement.
Note that '4 is an odd integer' is also a statement, but it is a false statement.
Are these sentences statements or not? If yes, are they true or false? Can you prove it?

- Telephone numbers in the USA have 10 digits.
- The moon is made of cheese.
- The sum of 2 odd natural number is an even number
- Would you like some cake?
- $3 + x = 12$
- The sum of two even numbers.
- $1 + 3 + 5 + 7 + \cdots + 2n + 1$.
- Go to your room!
- $7 + 3 = 10$
- All birds can fly.

The rule of divisibility by 2 is:
If the last digit of a number is an even number or 0 (0, 2, 4, 6, or 8) the number is even number (divisible by 2).

Divisibility Rules	
A number is divisible by	
2	If last digit is 0, 2, 4, 6, or 8
3	If the sum of the digits is divisible by 3
4	If the last two digits is divisible by 4
5	If the last digit is 0 or 5
6	If the number is divisible by 2 and 3
7	cross off last digit, double it and subtract. Repeat if you want. If new number is divisible by 7, the original number is divisible by 7
8	If last 3 digits is divisible by 8
9	If the sum of the digits is divisible by 9
10	If the last digit is 0
11	Subtract the last digit from the number formed by the remaining digits. If new number is divisible by 11, the original number is divisible by 11
12	If the number is divisible by 3 and 4

Proof of the divisibility by 2 rule:
Any natural number can be written as a sum:

$$\ldots + 1000 \cdot n + 100 \cdot m + 10 \cdot l + k = \cdots + 2 \cdot 500 \times n + 2 \cdot 50 \times m + 2 \cdot 5 \cdot l + k$$

Were n, m, l, and k are numbers of thousands, hundreds, tens, and units. If k is an even number or 0, it also can be represented as a product of 2 and another single digit number. Then the number can be written as:

$$\ldots + 1000 \times n + 100m + 10 \times l + k = \cdots + 2 \times 500 \times n + 2 \times 50 \times m + 2 \times 5 \times l + 2 \times p$$

(p can be 0, 1, 2, 3, and 4. Do you know why?). Distributive property is allowing us to represent this expression as a product:

$$\ldots + 1000 \times n + 100m + 10 \times l + k = \cdots + 2 \times 500 \times n + 2 \times 50 \times m + 2 \times 5 \times l + k$$
$$= 2 \times (\ldots + 500 \times n + 50 \times m + 5 \times l + p)$$

Now we can see that the number is divisible by 2 if its last digit is even or 0.
All other divisibility rules can be proved as well.

Factorization.

In mathematics factorization is a decomposition of one number into a product of two or more numbers, or representation of an expression as a product of 2 or more expressions, which called 'factors'. For example, we can represent the expression $a \cdot b + a \cdot c$ as a product of a and expression $(b + c)$. Can you explain why?

$$a \cdot b + a \cdot c = a \cdot (b + c)$$

Or in a numerical expression:

$$7 \cdot 5 + 7 \cdot 3 = 7 \cdot (5 + 3)$$

Or a number can be representing as product of two or more other numbers, for example:

$$40 = 4 \cdot 10 = 4 \cdot 2 \cdot 5, \quad 36 = 6 \cdot 6 = 3 \cdot 2 \cdot 6$$

Does any natural number can be represented as a product of 2 or more numbers besides 1 and itself? Natural numbers greater than 1 that has no positive divisors other than 1 and itself are called prime numbers.

Even numbers are the numbers divisible by 2 (they have 2 as a divisor), so they can be factorized as 2 times something else. Can an even number be a prime number? Is there any even prime number?

Prime factorization or integer **factorization** of a number is the determination of the set of **prime** numbers which multiply together to give the original integer. It is also known as **prime** decomposition.

168	*2*
84	*2*
42	*2*
21	*3*
7	*7*
1	

180	*2*
90	*2*
45	*3*
15	*3*
5	*5*
1	

Prime factorization process:

Prime factors of 168 are $2, 2, 2, 3, 7$ and prime factors of 180 are $2, 2, 3, 3, 5,$

$$2 \times 2 \times 2 \times 3 \times 7 = 168; \quad 2 \times 2 \times 3 \times 3 \times 5 = 180$$

Eratosthenes proposed a simple algorithm for finding prime numbers. This algorithm is known in

mathematics as the Sieve of Eratosthenes.

In mathematics, the sieve of Eratosthenes, one of a number of prime number sieves, is a simple, ancient algorithm for finding all prime numbers up to any given limit. It does so by iteratively marking as composite, *i.e.*, not prime, the multiples of each prime, starting with the multiples of 2.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Exercises:

1. Proof that the sum of two any even natural numbers is an even number.
2. The remainder of 1932 ÷ 17 is 11, the remainder of 261 ÷ 17 is 6. Is 2193 = 1932 + 261 divisible by 17? Can you tell without calculating? Explain.
3. Find all natural numbers such that upon division by 7 the quotient and remainder will be equal.
4. Even or odd number will be the sum and the product of
   a. 2 odd numbers
   b. 2 even numbers
   c. 1 even and 1 odd number
   d. 1 odd and 1 even number

   Can you explain why? (a few examples do not prove the statement).
5. Compute (what is the best way to compute it?):
   a. $23 \times 15 + 15 \times 77;$
   b. $79 \times 21 - 69 \times 21;$
   c. $340 \times 7 + 16 \times 70;$
   d. $250 \times 61 - 25 \times 390;$
   e. $67 \times 58 + 33 \times 58;$
   f. $55 \times 682 - 45 \times 682;$
6. Can you say which of the following statements are true and which are false?
   a. If the natural number is divisible by 4 and 3, it's divisible by 12 (if $a \vdots 3$ and $a \vdots 4 \Rightarrow a \vdots 12$)
   b. If the natural number is divisible by 12, it's divisible by 3 and 4. (if $a \vdots 12 \Rightarrow a \vdots 3$ and $a \vdots 4$)
   c. If $a \vdots 3$ then $a \vdots 9$.

d. If $a \vdots 9$ then $a \vdots 3$

7. Even or odd number will be the sum

$$1 + 2 + 3 + \ldots + 2020$$

8. Can the expression below be a true statement, if letters are replaced with numbers from 1 to 9 (different letters correspond to different numbers).

$$f \cdot l \cdot y = i \cdot n \cdot s \cdot e \cdot c \cdot t$$

6. $a + 1$ is divisible by 3. Prove that $4 + 7a$ is divisible by 3 as well.

7. $2 + a$ and $35 - b$ are both divisible by 11. Prove that $a + b$ is divisible by 11 as well.

8. Find all prime numbers p such that p and $5p + 1$ both are prime numbers.

9. Even or odd number will be the sum

$1 + 2 + 3 + \ldots + 10$

$1 + 2 + 3 + \ldots + 100$

$1 + 2 + 3 + \ldots + 100$

9. Set A is the set of numbers

A={372, 405, 700, 1075, 4399}

Find subsets of the set A

a. Multiples of 2
b. Multiples of 3
c. Multiples of 5

10. Evaluate:

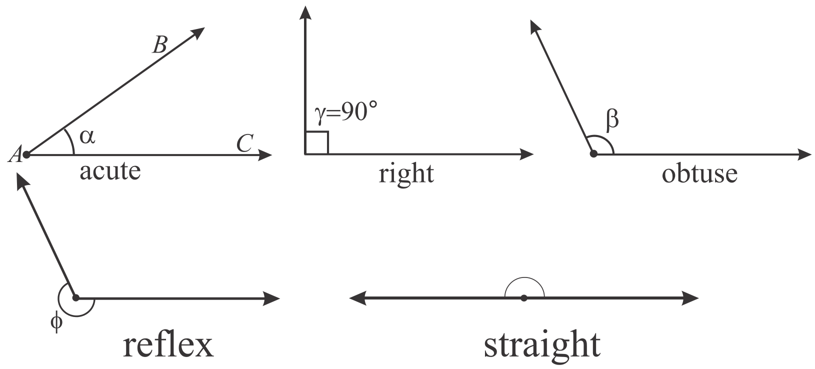
$$a. \quad \left(4\frac{1}{6} \cdot 3\right) : \left(7 \cdot \frac{5}{21}\right) - 1\frac{3}{4} \cdot 4 \qquad \left(answer: \frac{1}{2}\right);$$

$$b. \quad \left(4\frac{2}{5} + 3\frac{4}{5}\right) - \left(12 - 8\frac{1}{5}\right) \qquad \left(answer: 4\frac{2}{5}\right)$$

Geometry.

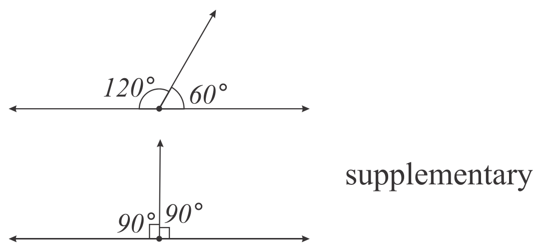
An angle is the figure formed by two **rays**, called the sides of the angle, sharing a common endpoint, called the **vertex** of the angle.

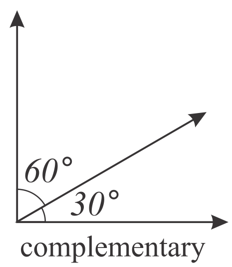
Angles notations are usually three capital letters with vertex letter in the middle or small Greek letter: ∠ABC, α. Measure of the angle is the amount of rotation required to move one side of the angle onto the other. As the angle increases, the name changes:



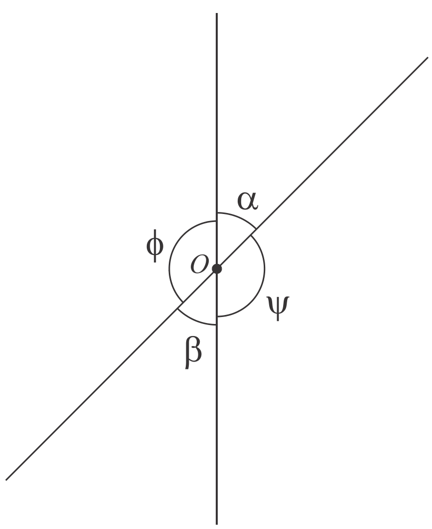
Strait angle is formed by two rays on the same strait line. Straight angle has a measure of 180°.

Two angles are called adjacent if they have common vertex and a common side. If two adjacent angles combined form straight angle they are called supplementary; if they form right angle than they are called complementary.





An angle which is supplementary to itself we call right angle. Lines which intersect with the right angle we call perpendicular to each other.



When two straight lines intersect at a point, four angles are formed. A pair of angles opposite each other formed by two intersecting straight lines that form an "X"-like shape, are called vertical angles, or opposite angles, or vertically opposite angles. α and β and ϕ and ψ are 2 pairs of vertical angles.

***Vertical angles theorem**:*

Vertical angles are equal.

In mathematics, a **theorem** is a statement that has been proven on the basis of previously established statements.

According to a historical legend, when Thales visited Egypt, he observed that whenever the Egyptians drew two intersecting lines, they would measure the vertical angles to make sure that they were equal. Thales concluded that one could prove that vertical angles are always equal and there is no need to measure them every time.

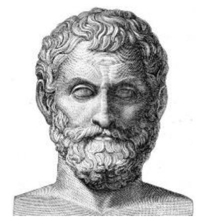
Proof:

$\angle\phi + \angle\alpha = 180°$ because they are supplementary by construction.

$\angle\phi + \angle\beta = 180°$ because they are supplementary also by construction.

$\Rightarrow \angle\alpha = \angle\beta$, therefore we proved that if 2 angles are vertical angles then they are equal. Can we tell that invers is also the truth? Can we tell that if 2 angles are equal than they are vertical angels?

(**Thales of Miletus** 624-546 BC was a Greek philosopher and mathematician from Miletus. Thales attempted to explain natural phenomena without reference to mythology. Thales used geometry to calculate the heights of pyramids and the distance of ships from the shore. He is the first known individual to use deductive reasoning applied to geometry, he also has been credited with the discovery of five theorems. He is the first known individual to whom a mathematical discovery has been attributed (Thales theorem).



Exercises.

11. Draw 3 different angles, measure them with a protractor.
12. Draw angles with the measures $72°, 155°, 90°$. Use ruler and protractor.
13. 4 angles are formed at the intersection of 2 lines. One of them is 30°. What is the measure of 3 others?
14. Draw 2 angles in such way that they intersect
    a. by a point
    b. by a segment
    c. by a ray
    d. don’t intersect at all.
15. Three lines intersect at 1 point and form 6 angles. One is 44°, another is 38°. Can you find all other angles? Draw the picture. Use protractor and ruler.
16. Right angle is divided into 3 angles by 2 rays. One angle by 20° more than the other and by 20° less the third one. What are the measures of these 3 angles?
17. On the picture below ∠BOD = 152°, ∠COD = 55°, angle ∠AOD is a straight angle. Find the measures of all other angles on the picture.
18. *On a line mark two points. How many segments were formed? Add one point. How many segments are there now? Add one more point. How many segments are there now? How many segments 6 points will form on the line? 10? 99?