MATH 9: VIETA AND TRIG

2021/03/21

1. Homework

1. Let x_1, x_2 and x_3 be distinct real numbers. Prove that there exists a unique polynomial, $p(x)$, of degree 2 such that $p(x_1) = 1$, $p(x_2) = p(x_3) = 0$. [Hint: if $p(x_1) = 0$, then $p(x)$ is divisible by $(x - x_1)$.] Find this polynomial if $x_1 = 2$, $x_2 = -1$, $x_3 = 5$.

2. Given any three ordered pairs of real numbers $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, prove that there exists a unique quadratic polynomial $p(x)$ such that $p(x_1) = y_1$, $p(x_2) = y_2)$, $p(x_3) = y_3$. Take a guess as to what you think a general result would be for n ordered pairs $(x_1, y_1), ..., (x_n, y_n)$.

3. Prove that if $p(x)$ is a polynomial with integer coefficients, then for any integers a, b, the difference $p(a) - p(b)$ is divisible by $a - b$.

4. Write Vieta formulas for the reduced cubic equation, $x^3 + px + q = 0$ (notice the coefficient of x^2 is 0). Let x_1, x_2, x_3 be the roots of this equation. Find the following combination in terms of p and q,

(a) $(x_1 + x_2 + x_3)^2$
(b) $(x_1)^2 + (x_2)^2 + (x_3)^2$
(c) $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
(d) $(x_1 + x_2 + x_3)^3$

5. Let the three real numbers x, y, z, satisfy the equations $x + y + z = 7$ and $1/x + 1/y + 1/z = 1/7$. Prove that then, at least one of x, y, z is equal to 7. [Hint: Vieta formulas]

6. Show that the length of a chord in a circle of unit diameter is equal to the sine of its inscribed angle.

7. Using the result of the previous problem, express the statement of the Ptolemy theorem in the trigonometric form, also known as Ptolemy identity:

$$\sin(\alpha + \beta)\sin(\beta + \gamma) = \sin\alpha\sin\gamma + \sin\beta\sin\delta$$

if $\alpha + \beta + \gamma + \delta = \pi$.

8. Prove the Ptolemy identity in the previous problem using the addition formulas for sine and cosine.

9. Using the Sine and the Cosine theorems, prove the Heron's formula for the area of a triangle,

$$S_{\triangle ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, c are the triangle's side lengths, and $s = \frac{a+b+c}{2}$ is the semiperimeter of the triangle.