Homework, assigned on March 14, 2021.

Algebra.

Read the classwork handout. Complete the unsolved problems from the previous homework. Solve the following problems. As usual, you do not necessarily have to solve every problem. However, please solve as much as you can in the time you have. Start with those that "catch your eye", in one way or another (e.g. you think are the easiest, or most challenging). Skip the ones you already solved in the past.

1. Perform long division of the following polynomials.
   a. $(x^5 - 2x^3 + 3x^2 - 4) \div (x^2 - x + 1)$
   b. $(x^4 - x^2 + 1) \div (x + 1)$
   c. $(x^7 + 1) \div (x^3 - x + 1)$
   d. $(6x^6 - 5x^5 + 4x^4 - 3x^3 + 2x - 1) \div (x^2 + 1)$
   e. $(x^5 - 32) \div (x + 2)$
   f. $(x^5 - 32) \div (x - 2)$
   g. $(x^6 + 64) \div (x^2 + 4)$
   h. $(x^6 + 64) \div (x^2 - 4)$
   i. $(x^{100} - 1) \div (x^2 - 1)$
2. Can you find coefficients a, b, such that there is no remainder upon division of a polynomial, $x^4 + ax^3 + bx^2 - 2x - 10$,
   a. by $x + 5$
   b. by $x^2 + x - 1$
3. Prove that,
   a. for odd n, the polynomial $x^n + 1$ is divisible by $x + 1$
   b. $2^{100} + 1$ is divisible by 17.
   c. $2^n + 1$ can only be prime if n is a power of 2 [Primes of this form are called Fermat primes; there are very few of them. How many can you find?]
   d. for any natural number n, $8^n - 1$ is divisible by 7.
   e. for any natural number n, $15^n + 6$ is divisible by 7
4. Factor (i.e., write as a product of polynomials of smaller degree) the following polynomials.

   a. $1 + x + x^2 + x^3$
   b. $1 - x + x^2 - x^3 + x^4 - x^5$
   c. $x^3 + 3x^2k + 3k^2x + k^3$

d. $a^3 + 3a^2b + 3b^2a + b^3$
e. $x^4 - 3x^2 + 2$

5. Simplify the following expressions using polynomial factorization.
   f. $\frac{x+y}{x} - \frac{x}{x-y} + \frac{y^2}{x^2-xy}$
   g. $\frac{x^6-1}{x^4+x^2+1}$
   h. $\frac{a^3-2a^2+5a+26}{a^3-5a^2+17a-13}$
6. Solve the following equations
   i. $\frac{x^2+1}{x} + \frac{x}{x^2+1} = 2.9$ (hint: substitution)
   j. $\frac{14}{20-6x-2x^2} + \frac{x^2+4x}{x^2+5x} - \frac{x+3}{x-2} + 3 = 0$ (hint: factorize square polynomials)
7. Write Vieta formulae for the reduced cubic equation, $x^3 + px + q = 0$. Let x_1, x_2 and x_3 be the roots of this equation. Find the following combination in terms of p and q,
   k. $(x_1 + x_2 + x_3)^2$
   l. ${x_1}^2 + {x_2}^2 + {x_3}^2$
   m. $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
   n. $(x_1 + x_2 + x_3)^3$
8. The three real numbers x, y, z, satisfy the equations

$$x + y + z = 7$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{7}$$

   Prove that then, at least one of x, y, z is equal to 7. [Hint: Vieta formulas]

9. Find all real roots of the following polynomial and factor it: $x^4 - x^3 + 5x^2 - x - 6$.