# MATH 9: POLYNOMIALS

2021/03/07

## 1. What is a Polynomial?

In algebra, we use variables, used in equations, functions, formulas, etc. A *monomial* is a variable to some exponent multiplied by some constant. Examples:

$$2x^2, \frac{1}{3}x^5, x^3, 1x^0$$

A polynomial is a sum of monomials. When monomials are added together, they can be simplified to have just one term for each different power of the variable, so we write them like this:

$$x^2 + x + 1$$

$$x^3 - \frac{1}{2}x$$

$$x^5 + 5$$

$$-x$$

$$7x^7 + 6x^6 + 5x^5 + 4x^4 + 3x^3 + 2x^2 + x$$

The number next to each power of $x$ is known as the *coefficient* of that power of $x$ in the polynomial. For example, the coefficient of $x^6$ in the last polynomial is 6. Coefficients are constants that do not depend on the value of $x$.

The highest exponent of $x$ found in a polynomial is known as its *degree*. The degree of the polynomials listed in the above list are 2, 3, 5, 1, 7.

Typically polynomials are written with a letter, as if they are a function, like $p(x)$ or $a(x)$. Indeed they can act like functions, you can plug a value in for $x$ and calculate the resulting value.

## 2. Polynomial Arithmetic

**Theorem 1** (Polynomial Arithmetic)**.** *When you add, subtract, or multiply two polynomials, you get a polynomial.*

This theorem is pretty straightforward, but it anticipates the following one:

**Theorem 2** (Polynomial Division)**.** *It is possible two divide two polynomials via a remainder with division process. Specifically, given polynomials $a(x)$ and $b(x)$, there exist polynomials $q(x)$ and $r(x)$, known as the quotient and remainder, such that $a(x) = q(x)b(x) + r(x)$, and the degree of $r$ is strictly less than the degree of $b$.*

*Proof.* The proof presented here is known as long division. A similar algorithm exists for dividing numbers to find quotient and remainder.
Suppose the highest-power term in $a(x)$ and $b(x)$ are $rx^m$ and $sx^n$, respectively. If $m < n$, then the degree of $a$ is less than the degree of $b$, so we take $q(x) = 0$ and $r(x) = a(x)$. If $m \geq n$, then let $q_0(x) = \frac{r}{s}x^{m-n}$. Now subtract $b(x)q_0(x)$ from $a(x)$. The highest exponent term will be wiped out, and you can repeat this process, and you keep going until your remainder finally has lower degree than $b(x)$.
This process can be written out using a specific drawing technique that you may have been taught in school for long division of numbers: write out the dividend $a(x)$, draw a line over it and a line to the left of it, write the divisor $a(x)$ to the left of the line, and then proceed by finding the highest terms of the quotient and writing them above the line, and subtract and repeat. $\square$

Note that the quotient and remainder are unique. This is apparent from the algorithm.

See the other uploaded document for an example of this algorithm carried out.

## 3. Polynomial Factorization

The first theorem presented here is known as the polynomial remainder theorem, or Little Bezout theorem. It relates polynomial arithmetic to polynomials as functions.

**Theorem 3** (Polynomial Remainder Theorem)**.** *Given a polynomial $p(x)$, and some number $t \in \mathbb{R}$, the remainder of $p(x)$ on division by $(x - t)$ is equal to $p(t)$.*

*Proof.* By the polynomial division theorem, there are polynomials $q(x)$ and $r(x)$ such that $p(x) = q(x)(x - t) + r(x)$, where the degree of $r(x)$ is less than the degree of $(x - t)$. But the degree of $(x - t)$ is 1, so the degree of $r(x)$ has to be zero. This means that $r(x)$ is a constant. This gives us $p(x) = q(x)(x - t) + r$. Plug in $x = t$ to get $p(t) = q(t)(t - t) + r = r$. $\square$

A *root* of a polynomial $p(x)$ is a number $t$ such that $p(t) = 0$. By the polynomial remainder theorem, this means that a root of a polynomial is a number $t$ such that $p(x)$ is divisible by $(x - t)$ with remainder 0. We can thus write $p(x)$ as $(x - t)q(x)$, for some quotient polynomial $q(x)$.

**Theorem 4** (Polynomial Factorization)**.** *Given a polynomial $p(x)$, if the distinct numbers $x_1$, ..., $x_k$ are roots of $p(x)$, then $p(x)$ is divisible by $(x - x_1)(x - x_2)...(x - x_k)$.*

*Proof.* First conclude that, because $x_1$ is a root of $p(x)$, that $p(x) = (x - x_1)q_1(x)$. But now if we plug in $x_2$, we get that $(x_2 - x_1)$ is nonzero, so $x_2$ must be a root of $q_1(x)$ in order for it to be a root of the whole thing. Therefore $x_2$ is a root of $q_1(x)$. Now apply the same logic to $q_1(x)$, to get $q_1(x) = (x - x_2)q_2(x)$. Repeat the same argument with $x_3$, ..., $x_k$, and you'll get the final result $p(x) = (x-x_1)(x-x_2)(x-x_3)...(x-x_k)q_k(x)$. $\square$

Breaking a polynomial down into a product of monomials that carry information about the roots is a particularly useful type of polynomial factorization. One final theorem is presented, that uses this result to prove something interesting.

**Theorem 5** (Number of Roots)**.** *Given $n$ distinct numbers, there is exactly one degree $n$ polynomial that has those $n$ numbers as roots. Additionally, there are no polynomials of degree less than $n$ that have all $n$ numbers as roots.*

*Proof.* Homework $\square$

## 4. Homework

**1.** When solving inversion problems, the goal usually is to turn as many circles into lines as possible. This means that the center of inversion is the most important detail, and then the radius can thence be chosen as desired. Typically, you pick a center of inversion such that the most useful circles go through that point. Remember that "go through" simply means that the point is on the circle.

Suppose you have two externally tangent circles, with radii 1 and 2 ("externally tangent" means the smaller circle is not inside the bigger circle). Prove that inversion around the point of tangency sends the circles to parallel lines. Then calculate the distance between the resulting lines as a function of the radius of the circle of inversion.

**2.** Suppose you have a rectangle $ABCD$ and a point $O$ inside the rectangle. Prove that inversion around $O$ sends the rectangle to a cyclic quadrilateral $A'B'C'D'$.

**3.** Let $\lambda$ be a circle with center $K$. Let $A$ be some point on $\lambda$.

(a) Prove that inversion around $A$ sends $\lambda$ and $\overleftrightarrow{AK}$ to perpendicular lines.
(b) Prove that $\lambda'$ is the perpendicular bisector of $\overline{A'K}$.

**4.** Let $A$ and $B$ be any two points.

(a) What is the locus of points $B'$ of reflections of $B$ around lines through $A$?
(b) Let $m$ be the line through $A$ perpendicular to $\overline{AB}$. What is the locus of points $B'$ of inversions of $B$ through circles with centers on $m$ that go through $A$? [Hint: if $O$ is on $m$, then $\angle OAB$ is a right angle. Where does this right angle end up after the inversion?]