MATH 9: SPRING REVIEW HOMEWORK

ASSIGNED 2021/02/07 ; DUE 2021/02/21

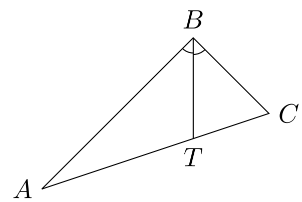
This homework sheet contains 12 problems, and you have two weeks to do them. There's a lot more here than you can do in one sitting though, so please plan your time wisely.

1. Problems done in class

This section contains problems or theorems that we discussed in class. For the algebra problems, I modified some of the numbers a little bit.

1. Complete, with proof, a straightedge-compass construction to solve the folowing problems. You must prove that your construction works.
   (a) Construct a circle through three given points.
   (b) Construct a circle tangent to three lines. (none of the lines are parallel)

2. Prove that the angle bisector of a triangle divides the opposite side in the same proportion as the adjoining sides.
In other words, let BT be the angle bisector of angle $\angle B$, prove that $\frac{AT}{TC} = \frac{AB}{BC}$.



3. Using Euclid's algorithm, provide the continued fraction representation for the following numbers.
   (a) 169/196
   (b) 89/55
   (c) 607/1650
   (d) 613/1666

4. Using mathematical induction, prove that
   (a) $\sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} = \frac{n}{3n+1}$
   (b) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
   (c) $\forall x > -1$ and $\forall n \geq 1$, $(1+x)^n \geq 1 + nx$
   (d) $\sum_{k=1}^{n} 2^k = 2^n - 1$ (this one was not done in class, but use your understanding of induction to solve it)

2. Logic and Set Theory

This section contains problems on logic and set theory. Take some time to review logic and wrap your head around these concepts.

1. Write out the truth tables for the following logical operations: $\wedge$, $\vee$, $\implies$, $\leftrightarrow$, $\neg$.

2. Describe set-builder notation, and explain its various parts. Then answer the following:
   (a) Explain what the set $\{z \in \mathbb{Z} | \frac{z}{2} \in \mathbb{Z}\}$ represents, and determine all points that are in this set.
   (b) Use set builder notation to define a subset $A \subset \mathbb{Z}$ such that A is infinite and $\mathbb{Z} \setminus A$ is also infinite.
   (c) Use set builder notation to define a subset B of $\mathbb{Z}$ such that for all integers n, the cardinality of $\{n-1, n, n+1\} \cap B$ is equal to 1.

3. Prove the union-intersection distributivity laws:
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (c) Now provide an example of three sets A, B, C such that $A \cup (B \cap C) \neq A \cap (B \cup C)$.

4. Prove that any infinite subset of $\mathbb{N}$ is countable.

5. Let the equivalence relation $\sim$ on $\mathbb{R}$ be given by integer-difference, i.e. $x \sim y$ is true if $(y - x) \in \mathbb{Z}$.
   (a) Describe the equivalence classes of this relation.
   (b) Prove that the set of equivalence classes is an uncountable set.

6. Provide an example of a countable collection of sets $C_0, C_1, C_2, \ldots$, each of which is a subset of $\mathbb{Z}$, such that $\forall n \in \mathbb{N}(\exists x \in \mathbb{Z}(x \in C_0 \cap C_1 \cap \ldots \cap C_n))$, but $\neg \exists x \in \mathbb{Z}(\forall n \in \mathbb{N}(x \in C_n))$.

3. Problems not done in class

This section contains two problems on combinatorics and algebra that were not discussed in class.

1. (a) Write out the inclusion-exclusion formula, and explain why it works.

(b) How many numbers are there from 1 to 999 whose digits contain at least one 1? What is the sum of all these numbers?

(c) Write out the formula for $\binom{n}{k}$, and explain why it works.

(d) How many numbers are there from 1 to 999 whose digits are in strictly decreasing order (i.e. each digit is less than the digit to its left)? What is the sum of all these numbers?

2. Recall the AM-GM inequality. State the inequality, and describe under what circumstances there is equality AM=GM. Then prove the following two mean comparison inequalities.

(a) HM-GM: $\forall a_1, ..., a_n$, $n \cdot (\frac{1}{a_1} + ... + \frac{1}{a_n})^{-1} \leq (a_1 \cdot ... \cdot a_n)^{1/n}$. HM means “harmonic mean”.

(b) AM-QM: $\forall a_1, ..., a_n$, $\frac{1}{n}(a_1 + ... + a_n) \leq \sqrt{\frac{1}{n}(a_1^2 + ... + a_n^2)}$. QM means “quadratic mean”.