MATH 9: MORE SET STUFF

2021/01/31

1. Classwork

We did these problems in class. yay!

1. (a) Construct a function $f : \mathbb{Z} \to \mathbb{Z}$ such that every element in the range of f is a perfect cube (an integer whose cube root is an integer). "Construct" a function means give the formula for the function, or some method of determining the output given any input. *The function $f(x) = x^3$ works.*

(b) Construct a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for each integer $n \in \mathbb{Z}$, there are exactly two integers in $\mathbb{Z}$ that map to n under f, i.e. there are exactly two integers x, y such that $f(x) = f(y) = n$). *The function $f(x) = floor(\frac{x}{2})$ works.*

2. (a) Given two functions $f : A \to B$ and $g : B \to C$ that are both bijections, prove that the function $g \circ f : A \to C$ is a bijection. *To prove this, prove that a composition of surjective functions is surjective, then prove that the composition of injective functions is injective. Now, we know that a function is bijective if and only if it is surjective and injective, therefore we can deduce that the composition of bijections is bijective.*

(b) Given two functions $f : A \to B$ and $g : C \to B$ that are both bijections, prove that there is a bijective function from A to C. *To solve this, use part (a). Since g is a bijection, it has an inverse, call it g^{-1}. Now apply part (a) to the composition $g^{-1} \circ f$ and this gives you a bijection from A to C.*

(c) Prove that if X is a countable set and $f : Y \to X$ is a bijection, then Y is also countable. *A set is countable if it has a bijection to $\mathbb{N}$. So we know that there is a bijection $g : X \to \mathbb{N}$. Now use part (a) to deduce the needed information about having a bijection from Y to $\mathbb{N}$.*

(d) Prove that, given any two countable sets X and Y, there is a bijective function from X to Y. *This is very similar to (c), but this time you need to use (b) instead of (a) to pull off the proof. In this case, X and Y both have bijections to $\mathbb{N}$, and this fact allows you to use part (b).*

2. Homework, More Set Stuff

1. (a) What is the range of $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x, y^2)$? What is the range of $f(x, y) = (x+y, x+y)$?

(b) Construct a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that the range of f is a line. Let the set F_0 be defined as $F_0 = \{(x, y) \in \mathbb{R}^2 \big| f(x, y) = (0, 0)\}$.

(c) Construct a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that the range of f is a line and the set F_0 is a line that is perpendicular to the range of f.

(d) Construct a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that the range of f is a line and the set F_0 is a line that is *not* perpendicular to the range of f.

2. Let $E \subset \mathbb{N}$ be the set of all even natural numbers and $O \subset \mathbb{N}$ the set of all odd natural numbers.

(a) Prove that E is countable. Do so by constructing a bijective function from E to $\mathbb{N}$.

(b) Prove that O is countable.

(c) Prove that $E \cup O$ is countable.

(d) Prove that, in general, given any two countable sets A, B, the union $A \cup B$ is countable.

3. Prove that any infinite subset of $\mathbb{N}$ is countable.

4. Let $T_n \subset \mathbb{N}$ be the set of all natural numbers whose power of 2 in their prime factorization is n. So for example, $2^2 3^5 5^1$ is in T_2, $2^3 11^2$ is in T_3, $2^0 3^1 5^2$ is in T_0, etc.

(a) List out the numbers from 1 to 20 and determine which of the sets T_n each number is in.

(b) Given any n, construct a bijection from T_n to T_{n+1}.

(c) Prove that T_0 is countable (you may use previous homework problems in this problem sheet as proof, if you have completed them).

(d) Prove that T_n is countable for all $n \in \mathbb{N}$.

(e) Prove that the union $T_0 \cup T_1 \cup T_2 \cup T_3 \cup ...$ is countable.

5. Let $f : \mathbb{N}^2 \to \mathbb{N}$ be given by $f(m, n) = (2m+1) \cdot 2^n$. First describe the range of f, then prove whether it is a bijection. Then explain whether you think $\mathbb{N}^2$ is countable.

6. Let A, B be countable sets. Prove that the product $A \times B$ is countable.

7. Let S be the set of all finite strings of letters and spaces. A string is any collection of characters, like "atxc rtt etet b", consisting of letters and spaces (regardless of whether it makes sense). Prove that S is countable.
Then prove that the set of sentences of the English language is countable.