MATH 9: CARDINALITY AND CONIC SECTIONS

2021/01/24

1. Homework, Sets

1. (a) Construct a function $f : \mathbb{Z} \to \mathbb{Z}$ such that every element in the range of f is a perfect cube (an integer whose cube root is an integer). "Construct" a function means give the formula for the function, or some method of determining the output given any input.
(b) Construct a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for each integer $n \in \mathbb{Z}$, there are exactly two integers in $\mathbb{Z}$ that map to n under f, i.e. there are exactly two integers x, y such that $f(x) = f(y) = n$).

2. (a) What is the range of $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x, y^2)$? What is the range of $f(x, y) = (x+y, x+y)$?
(b) Construct a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that the range of f is a line. Let the set F_0 be defined as $F_0 = \{(x, y) \in \mathbb{R}^2 \big| f(x, y) = (0, 0)\}$.
(c) Construct a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that the range of f is a line and the set F_0 is a line that is perpendicular to the range of f.
(d) Construct a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that the range of f is a line and the set F_0 is a line that is *not* perpendicular to the range of f.

3. (a) Given two functions $f : A \to B$ and $g : B \to C$ that are both bijections, prove that the function $g \circ f : A \to C$ is a bijection.
(b) Given two functions $f : A \to B$ and $g : C \to B$ that are both bijections, prove that there is a bijective function from A to C.
(c) Prove that if X is a countable set and $f : Y \to X$ is a bijection, then Y is also countable.
(d) Prove that, given any two countable sets X and Y, there is a bijective function from X to Y.

4. Let $E \subset \mathbb{N}$ be the set of all even natural numbers and $O \subset \mathbb{N}$ the set of all odd natural numbers.
(a) Prove that E is countable. Do so by constructing a bijective function from E to $\mathbb{N}$.
(b) Prove that O is countable.
(c) Prove that $E \cup O$ is countable.
(d) Prove that, in general, given any two countable sets A, B, the union $A \cup B$ is countable.

5. Prove that any infinite subset of $\mathbb{N}$ is countable.

2. Homework, Geometry

1. Draw four ellipses with eccentricities 0.2, 0.4, 0.6, 0.8. Compare the shapes of these ellipses. Which one looks most like a circle?
2. Given some line segment $\overline{AB}$, is it possible for two ellipses to have $\overline{AB}$ as their major axis but have different focal points?
3. Given a line l and a point F, what is the locus of points whose distance from F is twice the point's distance to l?
4. Given a right circular conic surface (see the images in the material sheet), what is the intersection of this surface with a plane that passes through the conic's vertex (the point where the two cones meet)? How does the result change if you change the angle of the plane of intersection with respect to the conic's axis?
5. One of the famous appearances of conic sections in physics is Kepler's laws. Look up Kepler's first law and understand what it says. Then answer this: what do you think the eccentricity of Earth's orbit is? Write down your guess, then look up the answer to check your guess.