Homework for November 22, 2020

Geometry.

Review the previous classwork notes. Solve the following problems, including problems from the last homework (if you have not solved them yet).

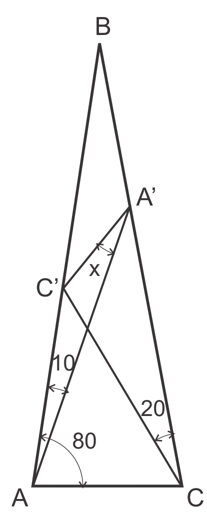
Problems.

1. Using the Ptolemy's theorem, prove the following:
    a. Given an equilateral triangle $\triangle ABC$ inscribed in a circle and a point Q on the circle, the distance from point Q to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.
    b. In a regular pentagon, the ratio of the length of a diagonal to the length of a side is the golden ratio, ϕ.
2. Given a circle of radius R, find the length of the sagitta (Latin for arrow) of the arc AB, which is the perpendicular distance CD from the arc's midpoint (C) to the chord AB across it.
3. Prove the Viviani's theorem:

    The sum of distances of a point P inside an equilateral triangle or on one of its sides, from the sides, equals the length of its altitude. Or, alternately,

    From a point P inside (or on a side) of an equilateral triangle ABC drop perpendiculars PP_a, PP_b, PP_c to its sides. The sum $|PP_a| + |PP_b| + |PP_c|$ is independent of P and is equal to any of the triangle's altitudes.

4. *In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 20°$ and $\angle BAA' = 10°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.



5. ** Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,

$$|AC| \cdot |BD| \leq |AB| \cdot |CD| + |BC| \cdot |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle (try using the triangle inequality).

Algebra.

Review the last classwork handout. Review and solve the classwork exercises which were not solved and unsolved problems from the previous homeworks.

1. Prove the following properties of the Cartesian product,
   a. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   b. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
   c. $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$
2. Find the Cartesian product, $A \times B$, of the following sets,
   a. $A = \{a, b\}, B = \{\uparrow, \downarrow\}$
   b. $A = \{June, July, August\}, B = \{1,15\}$
   c. $A = \emptyset, B = \{1,2,3,4,5,6,7,8,9\}$
3. Describe the set of points determined by the Cartesian product, $A \times B$, of the following sets (illustrate schematically on a graph),
   a. $A = [0,1], B = [0,1]$ (two segments from 0 to 1)
   b. $A = [-1,1], B = (-\infty, \infty)$
   c. $A = (-\infty, 0], B = [0, \infty)$
   d. $A = (-\infty, \infty), B = (-\infty, \infty)$
   e. $A = [0,1), B = \mathbb{Z}$ (set of all integers)
4. Propose 3 meaningful examples of a Cartesian product of two sets.
5. $n_A = |A|$ is the number of elements in a set A.
   a. What is the number of elements in a set $A \times A$
   b. What is the number of elements in a set $A \times (A \times A)$
6. Find the following sum.

$$\left(2 + \frac{1}{2}\right)^2 + \left(4 + \frac{1}{4}\right)^2 + \cdots + \left(2^n + \frac{1}{2^n}\right)^2$$

7. Find the following sum,
   a. $1 + 2 \cdot 3 + 3 \cdot 7 + \cdots + n \cdot (2^n - 1)$
   b. $1 \cdot 3 + 3 \cdot 9 + 5 \cdot 27 + \cdots + (2n - 1) \cdot 3^n$