November 15, 2020

Algebra.

Elements of Set Theory.

Definition. We will define a **set** to be a group of objects (not necessarily ordered) with no duplicates.

Note that the objects in the sets can themselves be sets. We can describe a set by defining some property of objects in it. For example,

1. the set containing the positive integers from 1 to 5 is $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$
2. the set of all natural numbers, which we denote $\mathbb{N}$
3. the set of all integer numbers, which we denote $\mathbb{Z}$
4. the set of all rational numbers, $\frac{\boldsymbol{m}}{\boldsymbol{n}}$, $(\{\boldsymbol{m}, \boldsymbol{n}\} \in \mathbb{Z} \wedge \boldsymbol{n} \neq \mathbf{0})$, which we denote $\mathbb{Q}$
5. the set of all real numbers, which we denote $\mathbb{R}$
6. the set of all irrational numbers, which we denote **D**

If a set has finite number of objects, it is said to be finite. Otherwise, it is infinite. The number of elements, $\boldsymbol{n}$, in a finite set $\boldsymbol{A}$, is denoted $|\boldsymbol{A}| = \boldsymbol{n}$. If elements in the set can be counted by assigning a natural number to each element, the set is called countable. The set that is not countable is called uncountable.

Exercise. Give examples of infinite, countable, uncountable sets.

If we wish to describe an infinite set, such as the set of even positive integers, we use what is called "set builder notation".

$M = \{x : (x \in \mathbb{Z}) \wedge (x > 0) \wedge (x/2 \in \mathbb{Z})\}$

This is read verbally as "the set of all x such that x is integer and greater than 0 and x divided by 2 is also integer". Another example,

$F = \{n^2 - 4 : (n \in \mathbb{Z}) \wedge (0 \leq n \leq 19)\}$

"F is the set of all numbers of the form $n^2 - 4$, such that n is a whole number in the range from 0 to 19 inclusive", where the colon ":" is read "such that".

If x is a member of a set M, we will use notation $x \in M$, if y is not a member of a set M we will write $y \notin M$. For example statement "$0 \leq x \leq 1$ " can be written as $x \in [0,1]$. Another example: If $x > 3$ or $x < 5$, so $x \in (-\infty, 5] \cap [3, +\infty)$.

Set builder notation can also be written like this:

$$M = \{x \in \mathbb{Z} : (\boldsymbol{x} > \mathbf{0}) \wedge (\boldsymbol{x} < \mathbf{2})\}$$

And also a vertical bar is often used instead of the colon, like this:

$$M = \{x \in \mathbb{Z} \,|(x > 0) \wedge (x < 2)\}$$

In some applications, you may want to modify the x that goes in the set. This is not generally common in theoretical mathematics, but finds interesting applications in programming for example. In any case, if this helps you understand set builder notation better, you can also think of it like this, in three parts:

$$M = \{f(x)|x \in K|P(x)\}$$

Which just means f(x) for x in some domain set K if the logical property P(x) holds true. For the sake of common mathematical notation, though, I will stick to writing $(x \in K) \wedge P(x)$ for the domain and property parts of the notation.

<u>The algebra of sets</u>.

An **algebraic structure** (algebra) is formed by a set of objects supplemented by a set of operations, which act on the elements of this set and obey certain algebraic laws. Typical example of an algebra are binary operations of addition and multiplication on a set of real, or integer numbers, which combine two elements to produce a third. These operations obey certain laws, such as commutative, associative, and distributive. Another example would be a set of all possible rotations of a solid body, with multiplication defined as combination of two consecutive rotations (Lie algebra, it is associative, but not commutative). The algebra of sets is an algebraic structure consisting of operations on sets (the elements of the set of sets).

Definition. An identity element with respect to a binary operation on a set is a set which leaves other elements unchanged when combined with them. An identity with respect to binary addition is called an additive identity (often denoted as 0) and an identity in the case of multiplication a multiplicative identity (often denoted as 1).

Definition. The **empty set** (or **null set**) is the set which contains no objects and is denoted {}, or by the symbol $\varnothing$.

Definition. The **universal set** I (the Universe of discourse) is the set which contains all objects of any nature, and of which all other sets are subsets.

In the algebra of sets, the empty set and the universal set play roles of the additive and the multiplicative identity, respectively.

Definition. The set A is said to be a **subset** of the set B if there is no element in A that is not also in B. It is denoted by $A \subset B$, or $B \supset A$.

Exercise. Let A be a finite set, with the number of elements $|A| = n$. How many different subsets does A have (including the empty subset and A itself)?

<u>Comparing sets.</u>

If both statement $A \subset B$ and $B \subset A$ hold, then sets A and B are equal, $A = B$. In this case sets A and B contain exactly the same elements. The relation $A \subset B$ has some similarities with the $a \leq b$ relation between the real numbers. In particular, the following <u>set comparison rules</u> hold:

1. $A \subset A$
2. If $A \subset B$ and $B \subset A$ then $A = B$
3. If $A \subset B$ and $B \subset C$ then $B \subset C$
4. $\emptyset \subset A$ for any set A
5. $A \subset \mathrm{I}$ for any set A

The difference between the order relation $A \subset B$ between sets and the $\leq$ relation between real numbers is that for numbers either $a \leq b$, or $a \geq b$ always holds, while this is not necessarily the case for sets order relation.

Definition. The **union** of two sets A and B is the set of elements, which are in A **or** in B **or** in both. It is denoted by $A \cup B$ and is read 'A union B'.

Definition. The **intersection** of two sets A and B is the set of elements, which are in A **and** in B. It is denoted by $A \cap B$ and is read 'A union B'.

We can associate the union with the "logical sum" of sets A and B,

$A \cup B = A + B$,

and the intersection with the "logical product",

$A \cap B \ = \ A \cdot B$.

Using these definitions, it can be easily verified that these operations satisfy the following rules.

6. $A + B = B + A$
7. $A \cdot B = B \cdot A$
8. $A + (B + C) = (A + B) + C$
9. $A \cdot (B \cdot C) = (A \cdot B) \cdot C$
10. $A + A = A$
11. $A \cdot A = A$
12. $A \cdot (B + C) = (A \cdot B + A \cdot C)$
13. $A + (B \cdot C) = (A + B) \cdot (A + C)$
14. $A + \varnothing = A$
15. $A \cdot I = A$
16. $A + \mathrm{I} = I$
17. $A \cdot \varnothing = \varnothing$
18. $A \subset B$ is equivalent to either of the two, $A + B = B$, or $A \cdot B = A$

Definition. The **complement** of set A in I is the set A', which consists of all objects in I which are not in A.

The operation of obtaining a complement A' has no analogs in the algebra of numbers, and possesses the following properties.

19. $A + A' = I$
20. $A \cdot \mathrm{A}' = \ \varnothing$
21. $\varnothing' \ = \ I$
22. $I' = \varnothing$
23. $A'' = A$
24. $(A \subset B) \Leftrightarrow (B' \subset A')$

25. $(A+B)' = A' \cdot B'$
26. $(A \cdot B)' = A' + B'$

These 26 laws of the algebra of sets possess an interesting duality symmetry: if we interchange $\subset$ and $\supset$, $+$ and $\cdot$, and $\varnothing$ and I, the same set of rules is obtained. Each of the 26 relations transforms in some other of these relations.

Exercise. Verify the above stated duality.

It is also remarkable from the point of view of the axiomatic constructions that all the above 26 laws, as well as all other theorems of set algebra can be deduced from the following three equation adopted as axioms, much like the Euclidian geometry.

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $(A' + B')' + (A' + B)' = A$

The operations $A \cdot B$ and $A \subset B$ are then defined by: $A \cdot B = (A' + B')'$ and $A \subset B$ means that $A + B = B$. The third relation can be rewritten as:
$A \cdot B + A \cdot B' = A$
Or, after deducing distributivity, $A \cdot (B + B') = A$

Exercise. Verify that all 26 rules of the set algebra can be obtained from the three axioms stated above.

Example. An algebraic structure satisfying all laws of the algebra of sets is provided by a set of eight numbers, $\{1,2,3,5,6,10,15,30\}$, where addition is identified with obtaining the least common multiple, multiplication with the greatest common divisor, $m \subset n$ to mean "m is a factor of n", and $n' = 30/n$,

- $m + n \equiv LCM(n, m)$
- $m \cdot n \equiv GCD(n, m)$
- $m \subset n \equiv (n = 0 mod(m))$
- $n' \equiv 30/n$.

Exercise. Verify that thus obtained algebra satisfies all rules of set algebra.

We observe that laws of the algebra of sets look similar to the laws of propositional logic and predicate calculus, if we identify

$A \cap B = A \cdot B$, with conjunction (AND), $A \wedge B$

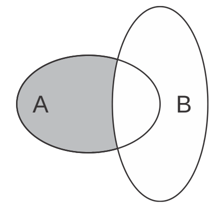
$A \cup B = A + B$, with disjunction (OR), $A \vee B$

A' with negation (NOT), $\sim A$

$A \supset B$ with $A \Rightarrow B$, $A \subset B$ with $A \Leftarrow B$.

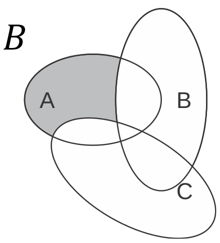
This is because any subset of a universal set can be defined using a predicate.

Definition. For two sets A, B, their difference $A - B$ (sometimes notation $A \backslash B$ is used instead of $A - B$) is defined by,



$$A - B \ = \ \{x \colon (x \in A) \wedge (x \notin B)\} = A \cap B'$$

The following properties can be shown to hold (consider Venn diagrams),



$A - (B \cup C) = \ (A - B) - C$, but in general, $A - (B - C) \neq (A \cup C) - B$

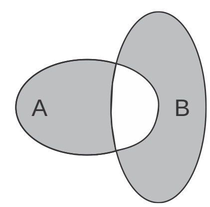
Because of this, although subtraction seems like an intuitive opposite of addition, it can't always be used to cleanly cancel out addition.

Exercise. Give an example of sets A, B where $A + (B - B) \neq (A + B) - B$

Definition. The symmetric difference of two sets is,

$$A \bigtriangleup B = (A - B) \cup (B - A)$$

This operation is commutative and associative,



$$A \bigtriangleup B = B \bigtriangleup A$$

$$(A \bigtriangleup B) \bigtriangleup C = A \bigtriangleup (B \bigtriangleup C)$$

The symmetric difference behaves uncannily like addition in important ways. It's the symmetric difference that's used to put a structure on sets that more closely resembles the algebra on numbers, i.e. arithmetic. All the same commutativity, distributivity, and identity laws hold for symmetric difference as addition and intersection as multiplication. But, with symmetric difference as addition, there *is* an exact opposite of the addition operation, so subtraction is possible in general equations. Can you describe what this subtraction operation would be?

Definition. For a set A, the characteristic function χ_A is defined as follows (the inputs are elements x),

$$\chi_A(x) = \begin{cases} 1, if\ x \in A \\ 0, if\ x\ \notin A \end{cases}$$

Exercise. Show that χ_A has following properties

$$\chi_A = 1 - \chi_{A'}$$

$$\chi_{A\cap B} = \chi_A\, \chi_B$$

$$\chi_{A\cup B} = 1 - \chi_{A'\cap B'} = 1 - \chi_{A'}\, \chi_{B'} = 1 - (1 - \chi_A)(1 - \chi_B) = \chi_A + \chi_B - \chi_A \chi_B$$

Exercise. Write a formula for $\chi_{A\cup B\cup C}$; $\chi_{A\cup B\cup C\cup D}$.

Solutions to some homework problems.

1. **Problem.** Write the first few terms in the following sequence ($n \geq 1$),

$$n\ fractions \left\{ \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}} \atop \quad \ldots + \frac{1}{1+x}} \right. = f_n$$

   a. Try guessing the general formula of this fraction for any n.

   b. Using mathematical induction, try proving the formula you guessed.

Solution. $n = 1$: $f_1 = \frac{1}{1+x}$; $n = 2$: $f_2 = \frac{1}{1+\frac{1}{1+x}} = \frac{1+x}{2+x}$; $n = 3$, $f_3 = \frac{1}{1+\frac{1}{1+\frac{1}{1+x}}} = \frac{2+x}{3+2x}$; $n = 4$, $f_4 = \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+x}}}} = \frac{3+2x}{5+3x}$; $f_5 = \frac{5+3x}{8+5x}$; … .

From the definition, we can write the recurrence, $f_{n+1} = \frac{1}{1+f_n}$. We note, that if $f_n = \frac{a_n+b_nx}{c_n+d_nx}$, then $f_{n+1} = \frac{c_n+d_nx}{(a_n+c_n)+(b_n+d_n)x}$. Hence, in each next term, f_{n+1}, in the sequence, the numerator is equal to the denominator of the previous term, f_n, while the numbers in the denominator are the sums of the corresponding numbers in the numerator and the denominator of the previous term, f_n, thus forming the Fibonacci sequence, $\{F_n\} = \{1,1,2,3,5,8,13, \ldots\}$. We can thus guess,

a. $n\ fractions$: $f_1 = \frac{1}{1+x}$, $f_n = \frac{F_n+F_{n-1}x}{F_{n+1}+F_nx}$, $n > 1$

b. <u>Base</u>: $f_2 = \frac{1+x}{1+2x}$

   <u>Induction</u>: Using the recurrence implied in the definition,

$$f_{n+1} = \frac{1}{1+f_n} = \frac{1}{1+\frac{F_n+F_{n-1}x}{F_{n+1}+F_nx}} = \frac{F_{n+1}+F_nx}{F_{n+1}+F_n+F_nx+F_{n-1}x} = \frac{F_{n+1}+F_nx}{F_{n+2}+F_{n+1}x}.$$

2. **Problem**. Can you prove that,

a.

$$\frac{3+\sqrt{17}}{2} = 3 + \frac{2}{3+\frac{2}{3+\frac{2}{3+\cdots}}}?$$

b. $1 = 3 - \frac{2}{3-\frac{2}{3-\frac{2}{3-\cdots}}}?$

c.

$$\frac{4}{2+\frac{4}{2+\frac{4}{2+\cdots}}} = 1 + \frac{1}{4+\frac{1}{4+\frac{1}{4+\cdots}}}?$$

Find these numbers?

Solution. Consider a general continued fraction,

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cdots}}}$$

<u>If a number exists, which is equal to the above infinite continued fraction</u>, then it must satisfy the equation, $x = a + \frac{b}{x} \Leftrightarrow x^2 - ax - b = 0$ $\Leftrightarrow x = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 + b}$. If a and b are positive, then x must also be positive, so $x = \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 + b}$.

a. Following the above argument with $a = 3$, $b = 2$, we obtain, $x = \frac{3}{2} + \sqrt{\left(\frac{3}{2}\right)^2 + 2} = \frac{3+\sqrt{17}}{2}$

b. In this case, $a = 3$, but $b = -2$ is negative. Applying the above considerations naively, we obtain, $x = 3 - \frac{2}{x} \Leftrightarrow x^2 - 3x + 2 = 0$ $\Leftrightarrow (x-1)(x-2) = 0$, i.e. there are two equally “legitimate” answers, $x = 1$, or $x = 2$. What this means, is that assumption that there exist unique number encoded by the given infinite continued fraction is wrong: there exist no such number! In fact, this can also be understood by looking at finite truncations approximating this continued fraction. If the continued fraction is truncated after subtracting 2 and before division by 3, then it is equal to 1,

$3-\frac{2}{3-2}=1,\ 3-\frac{2}{3-\frac{2}{3-2}}=1,\ \dots.$

If, on the other hand, the truncation is after division by 3 and before subtracting 2, then we obtain a sequence of numbers approaching 2,

$3-\frac{2}{3}=2\frac{1}{3},\ 3-\frac{2}{3-\frac{2}{3}}=2\frac{1}{7},\ 3-\frac{2}{3-\frac{2}{3-\frac{2}{3}}}=2\frac{1}{15},\ \dots.$

c. Denote

$$x=\frac{4}{2+\frac{4}{2+\frac{4}{2+\cdots}}}=\frac{4}{2+x}$$

Then, $x^2+2x-4=0 \Leftrightarrow x=-1\pm\frac{\sqrt{5}}{2}$, and $x>0$. Hence, $x=-1+\frac{\sqrt{5}}{2}$.

Similarly, denote

$$y=\frac{1}{4+\frac{1}{4+\frac{1}{4+\cdots}}}=\frac{1}{4+y}$$

Then, $y^2+4y-1=0 \Leftrightarrow y=-2\pm\frac{\sqrt{5}}{2}$, and $y>0$. Hence, $y=-2+\frac{\sqrt{5}}{2}$, and $1+y=-1+\frac{\sqrt{5}}{2}=x$.