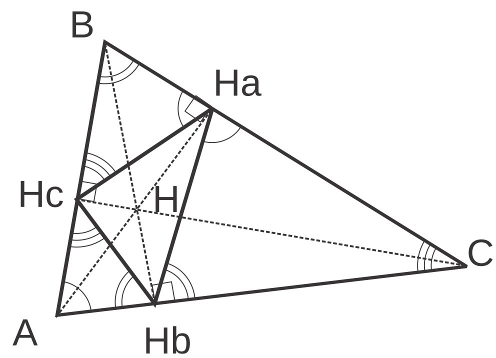
October 27, 2019

Geometry.

<u>Selected problems on similar triangles (from last homeworks)</u>.

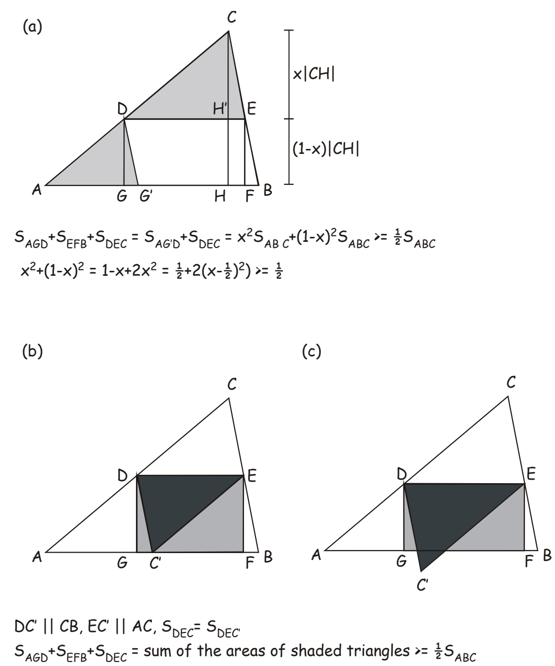
Problem 1(5). Prove that altitudes of any triangle are the bisectors in another triangle, whose vertices are the feet of these altitudes (hint: prove that the line connecting the feet of two altitudes of a triangle cuts off a triangle similar to it).



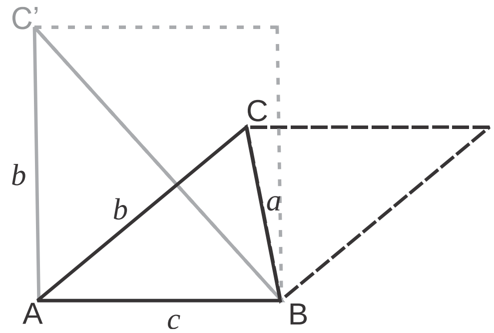
Solution. Notice similar right triangles, $ACH_a \sim BCH_b$, which implies, $\frac{|AC|}{|BC|} = \frac{|CH_a|}{|CH_b|}$. Therefore, $CH_aH_b \sim ABC$. Similarly, from $CAH_c \sim BAH_b$ it follows that $AH_bH_c \sim ABC$, and from $ABH_a \sim BCH_c$ that $BH_cH_a \sim ABC$.

Problem 2(2). Rectangle DEFG is inscribed in triangle ABC such that the side DE belongs to the base AB of the triangle, while points F and G belong to sides BC abd CA, respectively. What is the largest area of rectangle DEFG?

Solution. Notice similar triangles, $CDE \sim ABC$, wherefrom the vertical side of the rectangle is, $|DG| = |EF| = |CH| - |CH'| = \left(1 - \frac{|DE|}{|AB|}\right)|CH|$, so that the area of the rectangle is, $S_{DEFG} = |DE||DG| = |DE|\left(1 - \frac{|DE|}{|AB|}\right)|CH| = \frac{|DE|}{|AB|}\left(1 - \frac{|DE|}{|AB|}\right)|AB||CH| = \frac{|DE|}{|AB|}\left(1 - \frac{|DE|}{|AB|}\right)2S_{ABC}$. Using the geometric-arithmetic mean inequality, $\frac{|DE|}{|AB|}\left(1 - \frac{|DE|}{|AB|}\right) \le \left(\frac{\frac{|DE|}{|AB|}+1-\frac{|DE|}{|AB|}}{2}\right)^2 = \frac{1}{4}$, where the largest value of the left side is achieved when $\frac{|DE|}{|AB|} = 1 - \frac{|DE|}{|AB|}$, and therefore $S_{DEFG} = \frac{1}{2}S_{ABC}$. There are a number of other possible solutions, some of which are shown in the figures.

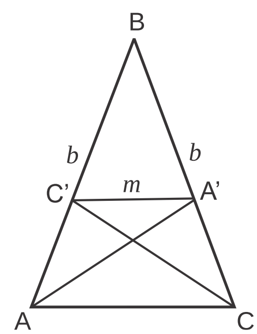


Problem 3(1). Prove that for any triangle ABC with sides a, b and c, the area, $S \leq \frac{1}{4}(b^2 + c^2)$.



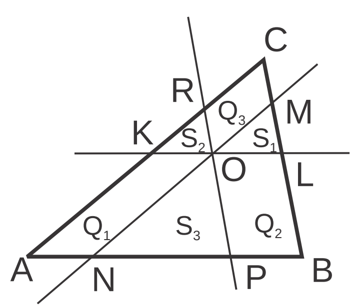
Solution. Notice that of all triangles with given two sides, b and c, the largest area has triangle ABC', where the sides with the given lengths, $|AB| = c$ and $|AC| = b$ form a right angle, $\widehat{BAC} = 90°$ (b is the largest possible altitude to side c). Therefore, $\forall \Delta ABC, S_{ABC} \leq S_{ABC'} = \frac{1}{2}bc \leq \frac{1}{2}\frac{b^2+c^2}{2}$, where the last inequality follows from the arithmetic-geometric mean inequality, $bc \leq \frac{b^2+c^2}{2}$ (or, alternatively, follows from $b^2 + c^2 - 2bc = (b-c)^2 \geq 0$.

Problem 4(2). In an isosceles triangle ABC with the side $|AB| = |BC| = b$, the segment $|A'C'| = m$ connects the intersection points of the bisectors, AA' and CC' of the angles at the base, AC, with the corresponding opposite sides, $A' \in BC$ and $C' \in AB$. Find the length of the base, $|AC|$ (express through given lengths, b and m).



Solution. From Thales proportionality theorem we have, $\frac{|AC|}{m} = \frac{|BC|}{|BA'|} = \frac{|BA'|+|A'C|}{|BA'|} = 1 + \frac{|A'C|}{|BA'|} = 1 + \frac{|AC|}{b}$, where we have used the property of the bisector, $\frac{|A'C|}{|BA'|} = \frac{|AC|}{|AB|} = \frac{|AC|}{b}$. We thus obtain, $|AC| = \frac{1}{\frac{1}{m}-\frac{1}{b}} = \frac{bm}{b-m}$.

Problem 5(5). Three lines parallel to the respective sides of the triangle ABC intersect at a single point, which lies inside this triangle. These lines split the triangle ABC into 6 parts, three of which are triangles with areas S_1, S_2, and S_3. Show that the area of the triangle ABC, $S = \left(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\right)^2$ (see Figure).



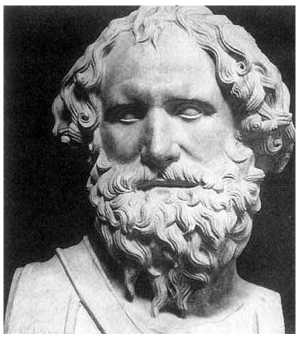
Solution. Denote $\frac{S_1}{S} = k_1, \frac{S_2}{S} = k_2, \frac{S_3}{S} = k_3$. Then, $\frac{S_1+S_2+Q_3}{S} = k_1 + k_2 + \frac{Q_3}{S} = \left(\sqrt{k_1} + \sqrt{k_2}\right)^2$, so, $Q_3 = 2S\sqrt{k_1k_2} = \sqrt{S_1S_2}$, $Q_2 = \sqrt{S_3S_1}$, $Q_1 = \sqrt{S_2S_3}$.

The Law of Lever. The Method of the Center of Mass.

Archimedes' Law of Lever.

"Give me a place to stand on, and I will move the earth."

quoted by Pappus of Alexandria in Synagoge, Book VIII, c. AD 340



Archimedes of Syracuse

Born c. 287 BC

Syracuse, Sicily

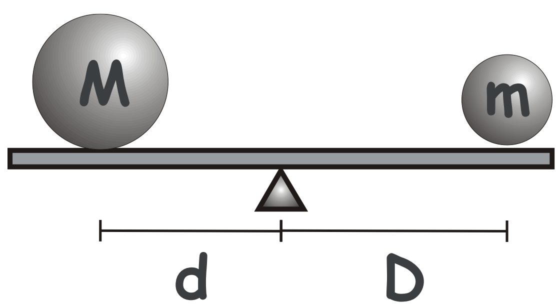
Magna Graecia

Died c. 212 BC (aged around 75) , Syracuse

Archimedes of Syracuse generally considered the greatest mathematician of antiquity and one of the greatest of all time. Archimedes anticipated modern calculus and analysis by applying concepts of infinitesimals and the method of exhaustion to derive and rigorously prove a range of geometrical theorems, including the area of a circle, the surface area and volume of a sphere, and the area under a parabola.

He was also one of the first to apply mathematics to physical phenomena, founding hydrostatics and statics, including an explanation of the principle of the lever. He is credited with designing innovative machines, such as his screw pump, compound pulleys, and defensive war machines to protect his native Syracuse from the Roman invasion.

Theorem (Law of Lever). Masses (weights) balance at distances from the fulcrum, which are inversely proportional to their magnitudes,



$$\frac{D}{d} = \frac{M}{m} \Leftrightarrow Md = mD$$

In terms of modern physics, it can be restated by saying that the weights balance if the torque on both sides is the same, where torque is defined as force*(distance to fulcrum).

Method of the Center of Mass (Mass Points).

Another way to restate Archimedes law of the lever is by using the notion of **center of mass.**

Definition. For two point masses, m_A and m_B at points A and B, the center of mass lies at a point C' on the straight line segment |AB| such that,

$$\frac{|AC'|}{|C'B|} = \frac{m_B}{m_A}.$$

Then we can restate Archimedes law by saying that the balance of the system will not change if we replace a pair of masses by a single mass $m_A + m_B$ placed at their center of mass. In particular, the system will be balanced if the center of mass is directly above the fulcrum.

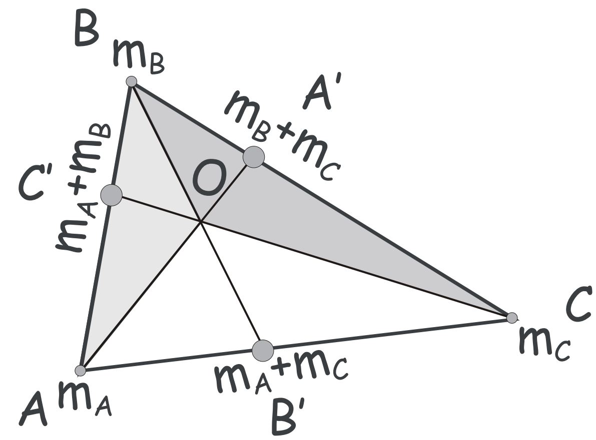
Center of mass can also be defined for a system of more than 2 point masses: by repeatedly replacing any pair of masses, m_A and m_B, with a single point mass having the total mass $m_A + m_B$, placed at the center of mass of the pair. (It is not obvious why the end result doesn't depend on the order we made these replacements – we will prove it later.)

So defined center of mass (COM) has the following important property:

> *The position of the system's center of mass does not change if we replace some subset of point masses with a single point mass, whose mass equals the sum of all these masses and which is positioned at their COM.*

<u>Ceva's Theorem: Formulation using COM.</u>

Place masses m_A, m_B, and m_C on vertices A, B, and C. The corresponding centers of mass for each pair are at points A', B' and C', respectively.



$$\frac{|AB'|}{|B'C|} \cdot \frac{|CA'|}{|A'B|} \cdot \frac{|BC''|}{|C'A|} = \frac{m_C}{m_A} \cdot \frac{m_B}{m_C} \cdot \frac{m_A}{m_B} = 1.$$

Then the three cevians meet at the center of mass of all three points. It is not hard to see that any three points must have a center of mass of the system (first collect two of the points into their center of mass on the segment connecting them, then finish up with the third point); Ceva's theorem is essentially the idea that this center of mass is the intersection of all three cevians.

To prove the other direction of Ceva's theorem using centers of mass, again it is not too difficult, use the fact that we proved one direction above and then do a proof by contradiction.