October 27, 2019

Algebra.

Arithmetic and geometric mean inequality: Proof by induction.

The **arithmetic mean** of n numbers, $\{a_1, a_2, \dots, a_n\}$, is, by definition,

$$A_n = \frac{a_1+a_2+\cdots+a_n}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i \qquad (1)$$

The **geometric mean** of n non-negative numbers, $\{a_n \geq 0\}$, is, by definition,

$$G_n = \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} = \sqrt[n]{\prod_{i=1}^{n} a_i} \qquad (2)$$

Theorem. For any set of n non-negative numbers, the arithmetic mean is not smaller than the geometric mean,

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} \qquad (3)$$

The standard proof of this fact by mathematical induction is given below.

Induction base. For $n = 1$ the statement is a true equality. We can also easily prove that it holds for $n = 2$. Indeed, $(a_1 + a_2)^2 - 4a_1a_2 = (a_1 - a_2)^2 \geq 0$ $\Rightarrow a_1 + a_2 \geq 2\sqrt{a_1a_2}$.

Induction step.

Assume that for some n, the inequality (3) holds for any set of n non-negative numbers, $\{a_1, a_2, \dots, a_n\}$. We want to prove that then, the inequality also holds for any set of $n + 1$ non-negative numbers, $\{a_1, a_2, \dots, a_{n+1}\}$.

Proof. If $a_1 = a_2 = \cdots = a_n = a_{n+1}$, then the equality obviously holds.

If not all numbers are equal, then there is the smallest (smaller than the mean) and the largest (larger than the mean). Reordering numbers if necessary, we can assume these two numbers are the last two ones, a_{n+1} and a_n respectively; thus, $a_{n+1} < A$, and $a_n > A$, where A is the arithmetic mean of the numbers a_1 through a_{n+1}. Consider new sequence of n

non-negative numbers, $\{a_1, a_2, \dots, a_{n-1}, a_n + a_{n+1} - A \}$. The arithmetic mean for these n numbers is still equal to A ,

$$\frac{a_1+a_2+\cdots+a_{n-1}+a_n+a_{n+1}-A}{n} = \frac{(n+1)A-A}{n} = A \tag{4}$$

Therefore, by induction hypothesis,

$$A^n \geq a_1 \cdot a_2 \cdot \dots \cdot a_{n-1} \cdot (a_n + a_{n+1} - A) \tag{5}$$

$$A^{n+1} \geq a_1 \cdot a_2 \cdot \dots \cdot a_{n-1} \cdot (a_n + a_{n+1} - A) \cdot A \tag{6}$$

It now remains to show that $a_n a_{n+1} < (a_n + a_{n+1} - A)A$,
$which\ easily\ follows\ from\ a_{n+1} < A$ and $a_n > A$. This completes the proof. ¤

<u>**Newton's binomial**</u>.

The **Newton's binomial** is a formula for $the\ polynomial\ (x+y)^n$ in the form of the polynomial summation (i.e. expanding the brackets),

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{k} x^{n-k} y^k + \cdots + \binom{n}{n-1} x\, y^{n-1} + \binom{n}{n} y^n, \tag{1a}$$

where we denote

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1b}$$

Other notations for the same quantity are C_n^k and nC_k

For $n = 1, 2, 3, \dots$, these are familiar expressions,

$$(x+y) = x + y,$$

$$(x+y)^2 = x^2 + 2xy + y^2,$$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$,

etc.

The Newton's binomial formula could be established either by directly expanding the brackets, or proven using the mathematical induction.

Exercise. Prove the Newton's binomial using the mathematical induction.

Induction basis. For $n = 1$ the statement is a true equality, $(x+y)^1 = C_1^0 x + C_1^1 y$. We can also easily prove that it holds for $n = 2$. Indeed, $(x+y)^2 = C_2^0 x^2 + C_2^1 xy + C_2^2 y^2$.

Induction hypothesis. Suppose the equality holds for some $n \in N$, that is,

$$(x+y)^n = C_n^0 x^n + C_n^1 x^{n-1} y + C_n^2 x^{n-2} y^2 + \cdots + C_n^k x^{n-k} y^k + \cdots + C_n^{n-1} x\, y^{n-1} + C_n^n y^n$$

Induction step. We have to prove that it then also holds for the next integer, $n+1$,

$$(x+y)^{n+1} = C_{n+1}^0 x^{n+1} + C_{n+1}^1 x^n y + C_{n+1}^2 x^{n-1} y^2 + \cdots + C_{n+1}^k x^{n+1-k} y^k + \cdots + C_{n+1}^n x\, y^n + C_{n+1}^{n+1} y^{n+1}$$

Proof. $(x+y)^{n+1} = (x+y)^n (x+y) =$

$$(C_n^0 x^n + C_n^1 x^{n-1} y + C_n^2 x^{n-2} y^2 + \cdots + C_n^k x^{n-k} y^k + \cdots + C_n^{n-1} x\, y^{n-1} + C_n^n y^n)(x+y) =$$

$$C_n^0 x^{n+1} + C_n^1 x^n y + C_n^2 x^{n-1} y^2 + \cdots + C_n^k x^{n-k+1} y^k + \cdots + C_n^{n-1} x^2\, y^{n-1} + C_n^n x y^n + C_n^0 x^n y + C_n^1 x^{n-1} y^2 + C_n^2 x^{n-2} y^3 + \cdots + C_n^k x^{n-k} y^{k+1} + \cdots + C_n^{n-1} x\, y^n + C_n^n y^{n+1} =$$

$$C_n^0 x^{n+1} + (C_n^1 + C_n^0) x^n y + (C_n^2 + C_n^1) x^{n-1} y^2 + \cdots + (C_n^k + C_n^{k-1}) x^{n-k+1} y^k + \cdots + (C_n^n + C_n^{n-1}) x\, y^n + C_n^n y^{n+1} =$$

$$C_{n+1}^0 x^{n+1} + C_{n+1}^1 x^n y + C_{n+1}^2 x^{n-1} y^2 + \cdots + C_{n+1}^k x^{n+1-k} y^k + \cdots + C_{n+1}^n x\, y^n + C_{n+1}^{n+1} y^{n+1},$$

Where we have used the property of binomial coefficients, $C_n^k + C_n^{k-1} = C_{n+1}^k$.
¤

Properties of binomial coefficients

Binomial coefficients are defined by

$$C_n^k = {}_nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial coefficients have clear and important combinatorial meaning.

- There are $\binom{n}{k}$ ways to choose k elements from a set of n elements.
- There are $\binom{n+k-1}{k}$ ways to choose k elements from a set of n if repetitions are allowed.
- There are $\binom{n+k}{k}$ strings of 0s and 1s containing k ones and n zeros.
- There are $\binom{n+1}{k}$ strings of 0s and 1s consisting of k ones and n zeros such that no two ones are adjacent.

They satisfy the following identities,

$$C_{n+1}^{k+1} = C_n^k + C_n^{k+1} \Leftrightarrow \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

$$C_{n+1}^{k} = C_n^k + C_n^{k-1} \Leftrightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\sum_{k=0}^{n} C_n^k = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Patterns in the Pascal triangle

Review of selected homework problems.

Problem 4. Using mathematical induction, prove that

a. $P_n \colon \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution.

Basis: $P_1: \sum_{k=1}^{1} k^2 = 1 = \frac{1\cdot(1+1)\cdot(2\cdot 1+1)}{6}$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1}: \sum_{k=1}^{n+1} k^2 = 1^2 + 2^2 + 3^2 + \cdots + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$

Proof: $\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)}{6}(n(2n+1) + 6n + 6) = \frac{(2n+1)(2n^2+7n+6)}{3} = \frac{(n+1)(n+2)(2n+3)}{6}$,

where we used the induction hypothesis, P_n, to replace the sum of the first n terms with a formula given by P_n. ¤

b. $P_n: \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Solution.

Basis: $P_1: \sum_{k=1}^{1} k^3 = 1 = \left[\frac{1(1+1)}{2}\right]^2$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1}: \sum_{k=1}^{n+1} k^3 = 1^3 + 2^3 + 3^3 + \cdots + (n+1)^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2$

Proof: $\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 = \left[\frac{(n+1)}{2}\right]^2 (n^2 + 4n + 4) = \left[\frac{(n+1)(n+2)}{2}\right]^2$, where we used the induction hypothesis, P_n, to replace the sum of the first n terms with a formula given by P_n. ¤

c. $P_n: \sum_{k=1}^{n} \frac{1}{k^2+k} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$

Solution.

Basis: $P_1: \sum_{k=1}^{1} \frac{1}{k^2+k} = \frac{1}{2} = \frac{1}{1+1}$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1}: \sum_{k=1}^{n+1} \frac{1}{k^2+k} = \frac{n+1}{n+2}$

Proof: $\sum_{k=1}^{n+1} \frac{1}{k^2+k} = \sum_{k=0}^{n} \frac{1}{k^2+k} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{n+1}{n+2}$. ¤

d. $P_n: \forall n \geq 2, \forall x > -1, \quad (1+x)^n \geq 1 + nx$

Solution.

Basis: $P_2: \forall x > -1, n = 2, \quad (1+x)^2 = 1 + 2x + x^2 \geq 1 + 2x$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1}: \forall n \geq 2, \forall x > -1, \ (1+x)^{n+1} \geq 1 + (n+1)x$

Proof: $(1+x)^{n+1} = (1+x)\,(1+x)^n \geq (1+x)(1+nx) = 1 + (n+1)x + x^2 \geq 1 + (n+1)x$. ¤