MATH 9: ASSIGNMENT 4

OCTOBER 18, 2020

1. Selected Solution

Hello! Recall we had the homework problem

$$\frac{x-a}{x-b} + \frac{x-b}{x-a} = 2.5$$

Here's the solution without paragraph explanations, if you prefer. See next page for the explanations.

$$\frac{x-a}{x-b} + \frac{x-b}{x-a} = 2.5$$
$$(x-b) \to (y-b)$$
$$\frac{x-a}{y-b} + \frac{y-b}{x-a} = 2.5$$
$$(x-a)^2 + (y-b)^2 = 2.5(x-a)(y-b)$$
$$(x-a) \to (u) \wedge (y-b) \to (v)$$
$$u^2 + v^2 = 2.5uv$$
$$(uv) \to (2)$$
$$(u^2 + v^2 = 5) \wedge (uv = 2)$$
$$(u^2 + 2uv + v^2 = 9) \wedge (uv = 2)$$
$$(u + v = 3) \wedge (uv = 2)$$
$$(u = 1) \wedge (v = 2)$$
$$v = 2u$$
$$(u) \to (x-a) \wedge (v) \to (y-b)$$
$$y - b = 2(x-a)$$
$$(y) \to (x)$$
$$x - b = 2(x-a)$$
$$x = 2a - b$$

Lastly, require $(x-a \neq 0) \wedge (x-b \neq 0)$, which simplifies to $a \neq b$.

Okay, the equation is

$$\frac{x-a}{x-b}+\frac{x-b}{x-a}=2.5$$

There are several ways to solve this, you can see the other sheet for Igor's solution, here is mine.

'Expand' this equation as if it's a two-dimensional plot, so replace $x-b$ with $y-b$. You then get the equation

$$\frac{x-a}{y-b}+\frac{y-b}{x-a}=2.5$$

Multiply both sides by the greatest common denominator of the fractions, which is $(x-a)(y-b)$

$$(x-a)^2+(y-b)^2=2.5(x-a)(y-b)$$

Each side of this looks familiar - the left hand side is a circle centered at (a,b), and the right hand side is a hyperbola centered at (a,b) with asymptotes parallel to the axes. Let's move the whole plot to the origin to make it easier to solve, and then we can move it back to finish up. In other words, let's use the substitution $u=x-a$ and $v=y-b$.

$$u^2+v^2=2.5uv$$

How do you solve this? Try drawing out the circle and hyperbola for a few values of uv, e.g. $uv=1$ and $uv=2$, you may notice that there are four intersection points. As uv is free to scale up and down, these four intersection points will move along two lines. These two lines are both through the origin. We can calculate their slope - to do so, all we need to find is one point on the lines (other than the origin), so let's take the most convenient value of uv, which happens to be 2 (you can play around with other values to see what happens if you want).

$$(u^2+v^2=5)\wedge(uv=2)$$
$$(u^2+2uv+v^2=9)\wedge(uv=2)$$
$$(u+v=3)\wedge(uv=2)$$

This last line is pretty easy to see that u,v are $1,2$ (in some order). You can use a quadratic equation to solve this for certain, but in general it's always good to guess one of the roots of a quadratic being 1 just to see if it works, which it does here.

$$(u=1)\wedge(v=2)$$

That means that one of the lines through the origin is $v=2u$. I'm not going to worry about the other line, you can see visually that there will only be two solutions to the final result, and since a,b are entirely symmetric in the original equation, it is enough to find one solution and then swap a,b to get the other solution. So, let's substitute x and y back in.

$$y-b=2(x-a)$$

And, finally, the last substitution we need is $x=y$, which returns us to the original one-variable equation where we took x and y to both be equal to x.

$$x-b=2(x-a)$$
$$x=2a-b$$

Swap a,b to get the other solution $x=2b-a$. Finally, qualify the result with the requirement that you cannot divide by zero in the original equation, so we don't want x to equal b or a; solving this out results in requiring that $a\neq b$.

2. Homework

1. Here is a rather famous problem if you have never seen it before:

You are given three toy houses and a stack of seven Matryoshka dolls (these are nesting dolls, you open one up to find a smaller one inside, which opens to find a smaller one, etc. - seven dolls means that they come in seven sizes, each one smaller than the last). Initially the dolls are all stacked inside each other and are inside the blue house. You want to move all the dolls to the green house. You are subject to several rules:

- Solitary rule: no house may appear to have more than one doll in it (e.g. if there are two dolls inside a house, one must be inside the other).
- Solo traveler rule: you are only allowed to move one doll at a time (you cannot move a whole stack - you must remove the outer doll and move just that one).
- Transience rule: if one doll is opened, you may not open another doll. This means that if you are moving a doll, you must place it into another house and close it before you open up any other dolls.
- Physical reality: you cannot put a bigger doll inside a smaller doll.

Prove that it is possible to move all the dolls from the blue house to the green house. (The third house is pink, you are allowed to use it too if it helps your process.)

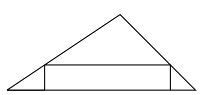
Then, determine and prove a formula for the minimum number of moves required to move the dolls from the blue house to the green house if you have n dolls instead of 7.

[This famous problem has a name, I will tell you next week. It is a classic example of proof by induction.]

2. Complete, with proof, a straightedge-compass construction to solve the folowing problems. State whether the solution to each problem is unique.

(a) Construct a circle through three given points.
(b) Construct a circle tangent to a line that goes through two given points. (the points are on the same side of the line)
(c) Construct a circle tangent to two lines that goes through a given point. (the lines are not parallel)
(d) Construct a circle tangent to three lines. (none of the lines are parallel)

3. You are given a fixed triangle with an area of 1. A rectangle is to be inscribed inside it so that its base is on the base of the triangle. What's the maximum possible area of the rectangle? ('Inscribed' means that all four vertices of the rectangle must be on the edges of the triangle.)



4. You are given two pterodactyl eggs and are climbing a multistory treehouse of n stories. You are told by the Brontosaurus Prefect that you must discover the maximum height at which pterodactyl eggs can be dropped without breaking. Assuming all pterodactyl eggs are the same, and that you are to give an answer in units of stories (e.g. "maximum height is fifth story of the tree"), what strategy can you come up with so as to minimize the number of times you need to drop an egg? Assume the worst case scenario (e.g. if your strategy is to try dropping an egg on each story one by one until you reach the top, climbing down to recover the egg after each drop, your worst case scenario would be to discover that it does not break until the top floor). If you break both eggs without discovering the correct height then you will be sentenced to extinction, so be careful.

5. Suppose you are given a Monty Hall problem: one prize, but n doors. Again, you select a door, and the host randomly opens one prizeless door. For which n is it better to switch doors than to stay to your original door?

6. (1984 USAMO) The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Find k.