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Algebra.

Formal fallacies (recap). Example of base rate fallacy: Monty Hall Problem.

A formal fallacy is an error in logic that can be seen in the argument's form. All formal fallacies are specific types of *non sequiturs* (does not follow).

- Base rate fallacy – making a probability judgment based on conditional probabilities, without accounting for the effect of prior probabilities.

Example. Consider playing the following game. There is a dollar bill in one of the three boxes. First, you are offered to choose one box. Then, the party you play with randomly chooses to open one of the remaining two boxes and you see that it is empty. You are then offered to swap your box and the remaining un-opened box. Should you switch? In other words, of the two un-opened boxes, the one you have and the unopened box from the other pair, which one has higher probability of containing the dollar?

This is a version of the famous problem in probability theory, also known as Bertrand box paradox, or an extensively discussed Monty Hall Problem, which was popularized by Martin Gardner as the Three Prisoners Problem.

Solution. An incorrect argument, which states that the probabilities of finding the dollar in the two un-opened boxes are equal because they were equal to begin with, before the third box was opened, is an example of base rate fallacy. Discarding the information on the probability of a condition "a box randomly chosen of the remaining two boxes is empty", yields the incorrect conclusion.

One simple solution is to count the possible outcomes: if the dollar is in the box you chose first, then there are two outcomes which satisfy the condition "a box randomly chosen of the remaining two boxes is empty". If, on the other hand, the dollar is in one of those boxes, then there is only one such outcome. Hence, after you know that one of the remaining two boxes, chosen at random is empty, you know that there are twice more chances that the dollar is in the box that remains unopened than in the one you chose first. Alternatively, there is one chance out of three that the dollar is in the box you chose, and two out of three that it is in one of the other two boxes. Once one of those two is opened and found empty, these two out of three chances now are that the dollar is in the unopened box. The solution can be formalized using the Bayes theorem on **conditional probability**.

Bayes' theorem (alternatively Bayes' law or Bayes' rule) describes the probability of an event, based on prior knowledge of conditions that might be related to the event,

$$P(A|B) = \frac{P(A \wedge B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

Here A and B are events whose probabilities without regard to each other are $P(A)$ and $P(B) \neq 0$.

- $P(A|B)$ is a conditional probability, the probability of observing event A given that B is true.
- $P(B|A)$ is the probability of observing event B given that A is true.
- $P(A \wedge B)$ is the probability of observing a joint event of both A and B.

Let A be the event "dollar is in the box A that you picked". $P(A) = \frac{1}{3}$ is the respective unconditional probability. The unconditional probability of the event B, "of the two remaining boxes, box B, chosen at random is empty", is, $P(B) = \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2}$. Note that here we have to consider events A and B as independent in order to obtain their probabilities without regard to each other. The probability that empty box B is opened when both B and C are empty, is $P(B|A) = \frac{1}{2}$, and we obtain,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{1}{3}$$

If C is now the event "dollar is in the remaining box C that has not been opened", then $P(C) = \frac{1}{3}$ and $P(B|C) = 1$, so

$$P(C|B) = \frac{P(B|C)P(C)}{P(B)} = \frac{2}{3}$$

Principle of Mathematical Induction (continued).

Let $\{P(n)\} = P(1), P(2), P(3), \ldots$ be a sequence of propositions numbered by positive integers, which together constitute the general theorem P. In particular, $P(n)$ can be some formula, or other property of positive integers.

Theorem (Principle of Mathematical Induction).

$$\big(P(1) \land (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1))\big) \Rightarrow (P: \forall n \in \mathbb{N}, P(n)).$$

Proof. Assume the opposite. Recalling that, $\sim(Q \Rightarrow P) \Leftrightarrow (Q \land \sim P)$, we write, the negation of the above statement as,

$$\big(P(1) \land (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1))\big) \land \sim(P: \forall n \in \mathbb{N}, P(n)),$$

Or,

$$\big(P(1) \land (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1))\big) \land (\exists n \in \mathbb{N}, \sim P(n)).$$

Now, using the "principle of smallest integer" we arrive at a contradiction,

$$\Big(\exists r \in \mathbb{N}, \big(P(r-1) \land \sim\big(P(r)\big)\big)\Big) \Leftrightarrow \sim(\forall r \in \mathbb{N}, P(r) \Rightarrow P(r+1)). \text{ ¤}$$

Example 1. Prove that the sum of the n first odd positive integers is n^2,

i.e., $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

Solution. Let $S(n) = 1 + 3 + 5 + \cdots + (2n-1)$.

We want to prove by induction that for every positive integer n, $S(n) = n^2$.

(1) Verify Base Case. For $n = 1$, we have $S(1) = 1 = 1^2$, so the property holds for $n = 1$.

(2) Inductive Step. Assume (Induction Hypothesis) that the property is true for a positive integer n, i.e.: $S(n) = n^2$. We must prove that it is also true for $n + 1$, i.e., $S(n+1) = (n+1)^2$, i. e., $\{S(n) = n^2\} \Rightarrow \{S(n+1) = (n+1)^2\}$. In fact, we can verify this explicitly,

$$S(n+1) = 1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = S(n) + (2n+1).$$

But, by induction hypothesis, $S(n) = n^2$. Hence,

$S(n+1) = n^2 + (2n+1) = (n+1)^2$.

This completes the induction and shows that the property is true for all positive integers. ¤

<u>Numerical sequences. Progressions.</u>

<u>**Numerical sequence**</u> is an ordered set of numbers, which are numbered consecutively by positive integers, n, $\{a_1, a_2, a_3, \dots, a_n\}$. The numbers, a_i, are called elements, or terms. The series is the value obtained by adding up all terms in the sequence; this value is called the "sum".

<u>**A Series**</u> is the sum of the terms of a sequence. Finite sequences and series have defined first and last terms, whereas infinite sequences and series continue indefinitely.

<u>**Arithmetic progression**</u> is the following numerical sequence,

$$\{a_1, a_2, a_3, \dots, a_n\} = \{a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \dots a_1 + (n-1)d\}. \quad (7)$$

The sum of the arithmetic progression is,

$$S_n = \frac{n}{2}(2a_1 + (n-1)d) = \frac{n}{2}(a_1 + a_n). \quad (8)$$

Excercise. Using mathematical induction, prove that

$$P_n : \sum_{k=1}^{n} k \ = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Solution.

<u>Basis</u>: $P_1 : \sum_{k=1}^{1} k \ = 1 = \frac{1 \cdot (1+1)}{2}$

<u>Induction</u>: $P_n \Rightarrow P_{n+1}$, where $P_{n+1} : \sum_{k=1}^{n+1} k \ = 1 + 2 + 3 + \cdots + (n+1) = \frac{(n+1)(n+2)}{2}$

Proof: $\sum_{k=1}^{n+1} k = 1 + 2 + 3 + \cdots + (n+1) = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$.

Geometic progression is a sequence,

$$\{a_1, a_2, a_3, \ldots a_n\} = \{a,\ aq^2, aq^3, \ldots, aq^{n-1}\} \qquad (9)$$

The sum of the geometric progression is geometric series,

$$G_n = a + aq + aq^2 + aq^3 + \cdots + aq^{n-1} = a\frac{1-q^n}{1-q}. \qquad (10)$$

This can be derived via several methods, including the mathematical induction.

Excercise. Using mathematical induction, prove formula for the sum of geometric series:

$$P_n: \sum_{k=0}^{n} q^k = 1 + q + q^2 + q^3 + \cdots + q^n = \frac{1-q^{n+1}}{1-q}$$

Solution.

Basis: $P_1: \sum_{k=0}^{1} q^k = 1 + q = \frac{1-q^2}{1-q}$

Induction: $P_n \Rightarrow P_{n+1}$, where $\sum_{k=0}^{n+1} q^k = 1 + q + q^2 + q^3 + \cdots + q^{n+1} = \frac{1-q^{n+2}}{1-q}$

Proof: $\sum_{k=1}^{n+1} q^k = 1 + q + q^2 + q^3 + \cdots + q^n + q^{n+1} = \frac{1-q^{n+1}}{1-q} + q^{n+1} = \frac{1-q^{n+1}+q^{n+1}-q^{n+2}}{1-q} = \frac{1-q^{n+2}}{1-q}$.

Examples. Using mathematical induction, prove that,

$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3-n}{3}$,

$2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2n(2n+1)(n+1)}{3}$.