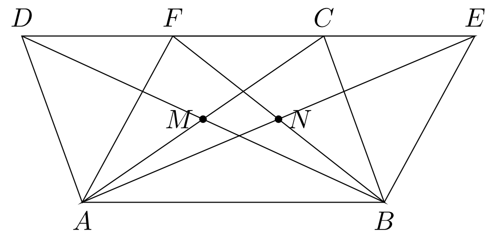
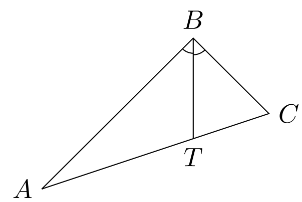
MATH 9: HOMEWORK 3

1. Geometry Problems

1. Let $ABCD$ and $ABEF$ be parallelograms such that E, F are on the line $\overleftrightarrow{CD}$; let the diagonals $\overline{AC}$, $\overline{BD}$ intersect at M and $\overline{AE}$, $\overline{BF}$ intersect at N. Prove that $\overline{MN} \parallel \overline{AB}$.



2. Prove that the angle bisector of a triangle divides the opposite side in the same proportion as the adjoining sides.
In other words, let BT be the angle bisector of angle $\angle B$, prove that $\frac{AT}{TC} = \frac{AB}{BC}$.



3. Complete (with proof) a straightedge-compass construction of a solution to the following problems. You may use without explanation construction of the following types (in addition to the usual circle and line): a line parallel to a given line through a given point; a line perpendicular to a given line through a given point; an angle congruent to a given angle through some other given ray.
  (a) A triangle whose side's midpoints are three given points.
  (b) A line tangent to two given circles. How many possible solutions are there?
  (c) A circle tangent to two given circles (where the given circles are themselves tangent to each other). How many possible solutions are there?

2. Algebra Problems

1. Solve the following equations for x.
  (a) $1 + \sqrt{1 + x\sqrt{x^2 + 8}} = x$
  (b) $\frac{x-1}{x+1} = y$
  (c) $\frac{x-a}{x-b} + \frac{x-b}{x-a} = 2.5$
  (d) $x - \frac{2}{x} - \frac{1}{x - \frac{2}{x}} = 0$

2. Please prove all of the following statements about rational numbers using only the properties of rational numbers discussed in class, together with the logical rules discussed in class. **Do not use any other properties about numbers.**
  (a) $\forall x(0 \cdot x = 0)$.
  (b) $\forall x(xx > 0)$. Here xx means $x \cdot x$, or x^2.
  (c) $x \cdot -1 = -x$. To prove this, prove that $x + (-1 \cdot x) = 0$.
  (d) Given $c < 0$, $a < b \leftrightarrow bc < ac$.
  (e) If 0 has a multiplicative inverse, then $0 = 1$.

3. Prove the following two logical statements.
  (a) $\forall x(\exists y(x \implies (y \implies x)))$. (Hint: try proof by contradiction)
  (b) $(((A \vee B) \implies A) \vee ((A \vee B) \implies B)) \wedge (A \implies (A \vee B)) \wedge (B \implies (A \vee B))$. (Hint: try to simplify first)

4. On the island of knights and knaves, every resident is either a knight or a knave; all knights always tell the truth, and all knaves always lie. Given a person named X, let $H(X)$ (the 'honor' of X) denote the logical statement 'X is a knight'. If X then tells you some logical statement, let $P(X)$

(the 'promise' of X) denote the statement that X says.
You are now faced with four people, named Celine, Bani, Pavel, and Zala, who tell you the following:

- Celine says that Bani would tell you that Pavel would tell you that Celine is a knight.
- Bani says that Pavel would tell you that Zala would tell you that Bani is a knight.
- Pavel says that Zala would tell you that Celine would tell you that Pavel is a knight.
- Zala says that Celine would tell you that Bani would tell you that Zala is a knight.

How many of them could possibly be knights? Is it possible that they're all knights? Is it possible that exactly one of them is a knight?

3. Additional Problem

A group of elves is famous for their ability to run in perfect synchronization with each other - in fact, they always run at exactly the same speed anytime they run. There is a local track in town, and out of the group of elves, any number of them may show up to the track on a particular day, during which they will space themselves evenly throughout the track and run clockwise at their standard speed.
On Sunday, seven elves show up to the track, as does Roshni; Roshni runs one lap counterclockwise. As Roshni is running opposite the direction of the elves, Roshni passes by an elf more than seven times: in fact, Roshni passes by an elf exactly nine times.
On Monday, four elves show up to the track, as does Ana; Ana runs one lap counterclockwise, passing elves eight times.
On Tuesday, eleven elves show up to the track, as does Jun; Jun runs one lap counterclockwise, passing elves sixteen times.
Rank Roshni, Ana, and Jun from fastest to slowest average running speed for the laps they ran.